UNIVERSITÄT ERFURT
Erziehungswissenschaftliche Fakultät

The 14th International Conference on Clifford Algebras and Their Applications in Mathematical Physics

7th – 11th September 2026
Erfurt, Germany

Conference proceedings

Local organisers:

Swanhild Bernstein

Sören Kraußhar

Anastasiia Legatiuk

Dmitrii Legatiuk

Supported by the German Research Foundation DFG

Preface

This conference is the 14th edition of the conferences series started in 1985 at the University of Canterbury, UK. The conference aims at bringing together leading experts and young researchers from the field of Clifford algebras and their various applications in mathematics, physics, engineering and other applied sciences.

The conference covers various topics related to Clifford algebras and their applications: mathematical physics, noncommutative geometry, differential geometry, Clifford analysis, robotics, signal processing, geometric algebra, applications of Clifford structures to machine learning/AI and computer vision, harmonic analysis, discrete structures, octonions and Cayley-Dickson algebras.

Contents

List of participants

1. Abdeljawad, Thabet, Prince Sultan University, Saudi Arabia
2. Almeida, Lucas, University of Campinas - UNICAMP, Brazil
3. Altavilla, Amedeo, University of Bari Aldo Moro, Italy
4. Amao, Pedro, Pontifical Catholic University of Peru, Peru
5. Bandara, Lashi, Deakin University, Burwood, Australia
6. Banouh, Hicham, University of Bouira, Algeria
7. Bär, Christian, University of Potsdam, Germany
8. Bernstein, Swanhild, TU Bergakademie Freiberg, Germany
9. Binosi, Giulio, University of Trento, Italy
10. Bisi, Cinzia, Ferrara University, Italy
11. Brdečková, Johanka, Brno University of Technology, Czech Republic
12. Brezov, Danail, University of Architecture, Civil Engineering and Geodesy, Bulgaria
13. Brunier, Jan, TU Darmstadt, Germany
14. Carbone, Antonio, Ferrara University, Italy
15. Cerejeiras, Paula, University of Aveiro, Portugal
16. Chatzakou, Marianna, Ghent University, Belgium
17. Colombo, Fabrizio, Polytechnic University of Milan, Italy
18. Conradt, Oliver, Goetheanum, Switzerland
19. Cordella, Davide, Marche Polytechnic University, Italy
20. De Almeida, Regina, University of Trás-os-Montes and Alto Douro, Portugal

21. De Febritiis, Chiara, Marche Polytechnic University, Italy
22. De Martino, Antonino, Polytechnic University of Milan, Italy
23. Dechant, Pierre-Philippe, University of Leeds, UK
24. Di Teodoro, Antonio, Deusto University, Spain
25. Diki, Kamal, Ghent University, Belgium
26. Ding, Chao, Anhui University, China
27. Dou, Xinyuan, Anhui University, China
28. El Haoui, Youssef, Moulay Ismail University, Meknès, Morocco
29. Ertem, Ümit, Ankara University, Turkey
30. Faustino, Nelson, University of Aveiro, Portugal
31. Fernandez, Arran, Eastern Mediterranean University, Turkey
32. Ferreira, Milton, Polytechnic University of Leiria, Portugal
33. Filimoshina, Ekaterina, HSE University, Russia
34. Furey, Cohl, Humboldt-Universität zu Berlin, Germany
35. Gentili, Graziano, University of Florence, Italy
36. Ghiloni, Riccardo, University of Trento, Italy
37. Gomes, Narciso, University of Cape Verde, Cape Verde
38. González Calvet, Ramon, Istitut Pere Calders, Spain
39. Hartmann, Angelo, University of Campinas - UNICAMP, Brazil
40. Hayhurst, Colin, University of Sussex, UK
41. Henselder, Peter, Hochschule Rhein-Waal, Germany
42. Hitzer, Eckhard, International Christian University, Japan
43. Hogan, Jeffrey, University of Newcastle, Australia
44. Hu, Ren, Great Bay University, China
45. Huo, Qinghai, Hefei University of Technology, China
46. Jeon, Wonju, KAIST, Republic of Korea

47. Kähler, Uwe, University of Aveiro, Portugal
48. Ketsaris, Nikolay, Israel
49. Kimsey, David, Newcastle University, UK
50. Köplinger, Jens, Deutsche Bank (TDI Cary), US
51. Kou, Kit Ian, University of Macau, China
52. Krasnov, Kirill, University of Nottingham, UK
53. Kraußhar, Rolf Sören, University of Erfurt, Germany
54. Krump, Lukáš, Charles University, Czech Republic
55. Krutov, Andrey, Charles University, Czech Republic
56. Krysl, Svatopluk, Charles University, Czech Republic
57. Lasenby, Anthony, Cambridge University, UK
58. Lavicka, Roman, Charles University, Czech Republic
59. Legatiuk, Anastasiia, University of Erfurt, Germany
60. Legatiuk, Dmitrii, University of Erfurt, Germany
61. Li, Hongbo, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, China
62. Lucas, Simão, Polytechnic University of Milan, Italy
63. Luna-Elizarraras, Maria Elena, Holon Institute of Technology, Israel
64. Malonek, Helmuth R., University of Aveiro, Portugal
65. Mantovani, Francesco, Polytechnic University of Milan, Italy
66. Marchuk, Nikolay, Steklov Mathematical Institute, Russia
67. Marques da Costa, João, Polytechnic University of Milan, Italy
68. McClellan, Gene, Applied Research Associates, Inc., US
69. Miller, William F., Rice University, US
70. Morreira, Pablo, Anáhuac Querétaro, Mexico
71. Mouayn, Zouhair, Université Sultan Moulay Slimane de Beni-Mellal, Morocco

72. Orelma, Heikki, Tampere University, Finland
73. Perotti, Alessandro, University of Trento, Italy
74. Pinos, A. Debernardi, Autonomous University of Barcelona, Spain
75. Pinton, Stefano, Polytechnic University of Milan, Italy
76. Prezelj, Jasna, University of Ljubljana, Slovenia
77. Riccardi, Federico, Polytechnic University of Turin, Italy
78. Richter, Wolf Dieter, University of Rostock, Germany
79. Richtsfeld, Alberto, Stockholm University, Sweden
80. Robertson, Phillip, RAND Corporation, US
81. Roelfs, Martin, University of Antwerp, Belgium
82. Rumyantseva, Sofia, HSE University, Russia
83. Sabadini, Irene, Polytechnic University of Milan, Italy
84. Sarfatti, Giulia, Marche Polytechnic University, Italy
85. Savi, Enrico, University of Angers, France
86. Schettini Gherardini, Tancredi, University of Bonn, Germany
87. Schneider, Baruch, University of Ostrava, Czech Republic
88. Shi, Haipan, Huzhou Normal University, China
89. Shi, Yun, Zhejiang University of Science and Technology, China
90. Shirokov, Dmitry, HSE University, Russia
91. Silva, Tomas, University of Campinas - UNICAMP, Brazil
92. Singh, Tejinder Pal, Tata Institute of Fundamental Research, India
93. Socroun, Thierry, Garfe, France
94. Soucek, Vladimir, Charles University, Czech Republic
95. Stoppato, Caterina, University of Florence, Italy
96. Suk, Julian, TU Munich, Germany
97. Tkachev, Vladimir, Linköping University, Sweden

98. Truong, Minh Huy, Vietnam
99. Turgay, Neşet Deniz, Eastern Mediterranean University, Turkey
100. Vanegas, Carmen Judith, Yachay University, Ecuador
101. Verlinden, Imke, University of Antwerp, Belgium
102. Vieira, Nelson, University of Aveiro, Portugal
103. Vlacci, Fabio, University of Trieste, Italy
104. Wang, Wei, Zhejiang University, China
105. Wirth, Jens, University of Stuttgart, Germany
106. Xu, Zhenghua, Hefei University of Technology, China
107. Zatloukal, Vaclav, Czech Technical University in Prague, Czech Republic
108. Zhang Yifan, University of Ostrava, Czech Republic
109. Zimmermann, Martha Lina, TU Bergakademie Freiberg, Germany

Abstracts

Plenary speakers

From curvature to Dirac eigenvalues and back

Christian Bär
University of Potsdam, Germany
christian.baer@uni-potsdam.de

In dimension two, the Gauss-Bonnet theorem provides a striking link between curvature and topology. In higher dimensions, scalar curvature is much more flexible and carries much less topological information - no direct analogue of Gauss-Bonnet is available. Spin geometry offers a different route: through the Lichnerowicz formula, scalar curvature controls the spectrum of the Dirac operator.
In this talk, we will see how this relation can also be used in the opposite direction. Gromov's hyperspherical radius measures how large a round sphere can be dominated by a manifold through a degree-nonzero Lipschitz map. An upper bound for the first Dirac eigenvalue will be presented in terms of this radius, with equality characterizing the round sphere. Combined with classically known lower eigenvalue estimates, this gives streamlined proofs of some rigidity results and obstructions to positive scalar curvature. This implies, in particular, the famous positive mass theorem.
The underlying argument combines spin geometry, index theory and, in odd dimensions, spectral flow. The talk is based on the paper [1].

References

[1] C. Bär, *Dirac eigenvalues and the hyperspherical radius*, J. Europ. Math. Soc., online first, 2026.

Special values of modular functions and some applications

Jan Bruinier
TU Darmstadt, Germany
bruinier@mathematik.tu-darmstadt.de

The values of modular functions at complex multiplication (CM) points play an important role in number theory. For instance, the values of the classical Klein j-invariant at CM points, also known as singular moduli, generate Hilbert class fields of imaginary quadratic fields. They are key ingredients in the explicit class field theory of imaginary quadratic fields.
A celebrated result of Gross and Zagier gives an explicit formula for the prime factorization of the norm of the difference of two singular moduli. Surprisingly, although such norms are rather large numbers, their prime factors turn out to be very small. A new approach to this result and generalizations to modular functions on Shimura varieties associated to orthogonal groups was found by Schofer, as well as in joint work of Kudla, Yang, and the speaker. It relies in a crucial way on the arithmetic properties of Clifford algebras of rational quadratic spaces, and on Borcherds' theory of automorphic products. In our lecture we give a brief overview of these results. Moreover, we report on recent joint work with Yang and Yu, in which we apply Schofer's formula, combined with other arguments, to study reductions of genus two curves with complex multiplication, generalizing results of Garcia, Ritzenthaler, and Rodríguez Villegas.

References

[1] R. Borcherds, *Automorphic forms with singularities on Grassmanians*, Invent. Math. 132, 491-562, 1998.

[2] J. Bruinier, S. Kudla, T. Yang, *Special values of Green functions at big CM points*, Int. Math. Res. Not. 2012:9, 1917-1967, 2012.

[3] E. Garcia, C. Ritzenthaler, F. R. Villegas, *An arithmetic intersection for squares of elliptic curves with complex multiplication*, arXiv:2412.08738, 2024.

[4] B. Gross, D. Zagier, *On singular moduli*, J. Reine Angew. Math. 355, 191-220, 1985.

[5] J. Schofer, *Borcherds forms and generalizations of singular moduli*, J. Reine Angew. Math. 629, 1-36, 2009.

From complex to hypercomplex Fock spaces

Kamal Diki
Ghent University, Belgium
kamal.diki@ugent.be

The Fock space is a fundamental example of a reproducing kernel Hilbert space with important applications in quantum mechanics, time–frequency analysis, mathematical analysis, and stochastic processes. In recent years, there has been increasing interest in developing quaternionic and Clifford counterparts of the classical Fock space and related topics. In this talk, I will discuss the extension of Fock space theory from complex to hypercomplex analysis. In particular, I will present several constructions introduced in recent years in the setting of slice hyperholomorphic, slice polyanalytic, Fueter regular, and poly-Fueter regular functions of a quaternionic variable. I will explain how these different quaternionic Fock spaces are connected via the Fueter mapping theorem and its polyanalytic extensions. I will conclude by presenting the role of the Clifford–Appell polynomials and their corresponding poly-Fueter extensions in the study of hypercomplex Fock spaces.

Clifford translates, wavelets, prolates, splines and bandpass bases

Jeffrey Hogan, Hamed Baghal Ghaffari, Neil Dizon, Joseph Lakey, Peter Massopust
University of Newcastle, Australia
jeff.hogan@newcastle.edu.au

When searching for wavelets and scaling functions in high dimensions, the design criteria often include compact support, regularity, orthogonality and a multiresolution structure. Other criteria can include:

(a) *non-separability*, i.e., they are not tensor products of functions of each coordinate; and

(b) *(approximate) rotation-covariance*, i.e., if the wavelet transform $W_\psi f$ of a signal f and the wavelet transform $W_\psi R_\sigma f$ of its rotation $R_\sigma f$ are computed, then the values of $W_\psi R_\sigma f$ are computable in a simple way from the values of $W_\psi f$ and knowledge of the rotation σ.

Criterion (a) is motivated by the fact that orthogonal separable wavelets can produce spurious artefacts in the direction of the coordinate axes when employed in compression algorithms and do not approximate oblique orientations well [1]. The motivations for criterion (b) are twofold. Firstly, the transforms implemented by rotationally covariant wavelets are *steerable* and can provide rich information about local orientation structure [2]. Secondly, the cost of training a neural network can be reduced by imposing on the network a convolutional structure (for translation-invariance) that is also rotationally covariant [3]. When wavelet transforms are employed as the initialisation step of such a network, the full network retains its covariance only when the wavelet itself enjoys rotation-covariance.
Standard constructions of wavelets in higher dimensions employ translations along rectangular grids. This biases the constructed wavelets toward separability and away from rotation-covariance. In this talk we discuss the a construction of non-separable, orthogonal, compactly supported, quaternion-valued wavelets on the plane using standard translations [4], [5], and constructions of Clifford-valued splines [6], [7], [8], wavelets, prolates [9], [10] and bandpass bases [11] using a different notion of translation arising naturally from the foundational operators of Clifford analysis.

References

[1] M. Lai, D.W. Roach. *Nonseparable symmetric wavelets with short support.* In: Wavelet Applications in Signal and Image Processing VII, 3813, SPIE, 1999.

[2] M. Unser, D. Van de Ville, *Wavelet steerability and the higher order Riesz transforms*, IEEE Image Proc, 19, 2010.

[3] D. Ruhe, J. Branstetter, P. Forré. *Clifford group equivariant neural networks*, NEURIPS, 2023.

[4] N.Dizon, J.A. Hogan. *An optimisation approach to non-separablequaternion-valued wavelet constructions*, arXiv:2311.12614, 2026.

[5] N.Dizon, J.A. Hogan. *Holistic processing of color images using novel quaternion-valued wavelets on the plane: a promising transformative tool*, IEEE Signal Processing Magazine, 41, 2024.

[6] J.A. Hogan, P. Massopust. *Quaternionic B-splines*, Journal of Approximation Theory, 224, 2017.

[7] J.A. Hogan, P. Massopust. *Quaternionic cardinal fundamental splines: interpolation and sampling*, Complex Analysis and Operator Theory, 13, 2019.

[8] J.A. Hogan, P. Massopust. *Clifford-Valued B-Splines.* In: N. Faustino, M. Ferreira, U. Kähler, N. Vieira (eds) Hypercomplex Analysis and Its Applications. ICHAA 2024. Trends in Mathematics, 9. Birkhäuser, 2025.

[9] H. Baghal Ghaffari, J.A. Hogan, J.D. Lakey. *New properties of Clifford prolate spheroidal wavefunctions*, Proceedings of SampTA, 2023.

[10] H. Baghal Ghaffari, J.A. Hogan, J.D. Lakey. *Clifford prolate spheroidal wave functions and associated shift frames.* In: Sampling, Approximation, and Signal Analysis – Harmonic Analysis in the Spirit of J. Rowland Higgins, eds. S.D. Casey, M.M. Dodson, P.J.S.G. Ferreira, A. Zayed, Birkhäuser, 2023.

[11] J.A. Hogan, J.D. Lakey. *Pseudo Clifford bandpass prolates*, Proceedings of SampTA, 2023.

The geometric analytic signal: Unifying harmonic conjugates, Clifford algebras, and robust denoising in 8D spaces

Yujie Zhou, Xuan Chen, Kit Ian Kou
University of Macau, China
kikou@um.edu.mo

For over a century, the classical analytic signal has served as a cornerstone of signal processing, precisely because its harmonic conjugate structure—intimately tied to the Poisson kernel—confers an innate resilience to noise that conventional amplitude-only methods lack. Yet, as we push into the era of 3D volumetric imaging, vector sensor arrays, and geometric data, we face a fundamental question: *How do we carry this elegant property into the hypercomplex domain without sacrificing the rich geometric relationships between multivector components?*

In this talk, we present our journey toward answering this question through the **Clifford-analytic signal**—a framework rooted in the 8-dimensional Clifford biquaternion algebra $\mathrm{Cl}(0,3)$. The core breakthrough is a carefully designed spectral restriction operator in the Clifford Fourier domain that generalizes the classical Hilbert transform pair into a superposition of geometric Hilbert transforms. Crucially, this construction preserves the complete phase-geometry coupling among signal components—a feature that component-wise processing invariably destroys.

Why does this matter practically? Poisson noise, ubiquitous in photon-limited imaging systems, corrupts not just intensity but the very geometric structure of multivector fields. Our framework yields a principled denoising algorithm that separates coherent geometry from stochastic noise without introducing phase distortions or spatial artefacts. We have rigorously validated this approach across diverse volumetric and temporal imaging modalities, where it consistently outperforms established baselines in both quantitative metrics and visual fidelity.

Ultimately, this work is more than a denoising solution; it represents a foundational bridge between classical harmonic analysis and Clifford geometric algebra. By reimagining the analytic signal through a higher-dimensional geometric lens, we open a new spectral frontier for processing non-Euclidean and multivectorial data—inviting the community to rethink how we handle signal, geometry, and noise in unison.

References

[1] Y. Yang, K. I. Kou, C. Zou, *Edge detection methods based on modified differential phase congruency of monogenic signal*, Multidim Syst Sign Process 29, 339–359, 2018.

[2] Z. Cai, K. I. Kou, *Three-dimensional generalized analytic signal associated with linear canonical transform in Clifford biquaternion domain*, Math. Meth. Appl. Sci. 46, 12675-12689, 2023.

[3] X. Hu, D. Cheng, K. I. Kou, *Convolution theorems associated with quaternion linear canonical transform and applications*, Signal Processing 202, 108743, 2023.

Exceptional geometry, Spin groups, and the standard model

Kirill Krasnov
University of Nottingham, UK
kirill.krasnov@nottingham.ac.uk

I will review recent progress in understanding how the Standard Model gauge group,

$$SU(3) \times SU(2) \times U(1)/\mathbb{Z}_6,$$

emerges from the groups Spin(9) and Spin(10), together with the related exceptional structures, including the Jordan algebra $h_3(\mathbb{O})$ and the bi-Cayley Hermitian Jordan triple system. This line of research was initiated by the work of Todorov and Dubois-Violette about a decade ago and is rooted in the exceptional geometry of the octonions. A central role is played by the family of complex structures on the octonions, parametrised by the choice of a unit imaginary octonion.

References

[1] M. Dubois-Violette, *Exceptional quantum geometry and particle physics*, Nucl. Phys. B, 912, 426-449, 2016.

[2] I. Todorov, M. Dubois-Violette, *Deducing the symmetry of the standard model from the automorphism and structure groups of the exceptional Jordan algebra*, Int. J. Mod. Phys. A, 33, no.20, 1850118, 2018.

[3] K. Krasnov, *$SO(9)$ characterization of the standard model gauge group*, J. Math. Phys., 62, no.2, 021703, 2021.

[4] L. Boyle, *The standard model, the exceptional Jordan algebra, and triality*, J. Math. Phys., 67, no.7, 071701, 2026.

[5] K. Krasnov, *Geometry of Spin(10) symmetry breaking*, J. Math. Phys., 65, no.8, 082302, 2024.

[6] J. C. Baez, E. Bokor, L. Boyle, *Jordan Pair Quantum Theory and the Standard Model*, arXiv:2607.10833, 2026.

Fractional Clifford analysis and hypercomplex operator calculus

Nelson Vieira
University of Aveiro, Portugal
nvieira@ua.pt

Fractional Clifford analysis extends classical Clifford analysis by combining higher-dimensional function theory with fractional calculus. A foundational contribution to the field was the work of Kähler and Vieira, which introduced a fractional function theory in higher dimensions through a fractional correspondence with the Weyl relations. Since then, the theory has developed into a flexible framework for fractional Dirac-type and Laplace-type operators.
In this talk, I will discuss a hypercomplex operator calculus for the fractional Helmholtz equation, with emphasis on factorization results, fundamental solutions, Borel-Pompeiu type formulas, and Hodge-type decompositions. I will also explain how classical and previously studied fractional cases are recovered as limiting situations, highlighting the role of this framework within our broader work on fractional Laplace and Dirac operators. Finally, I will indicate how the stationary theory connects with the non-stationary side of the subject through time-fractional parabolic Dirac operators and related Borel-Pompeiu formulas.

References

[1] U. Kähler, N. Vieira. *Fractional Clifford analysis.* In: Hypercomplex Analysis: New Perpectives and Applications, Trends in Mathematics Series, Eds: S. Bernstein, U. Kähler, I. Sabadini and F. Sommen, Birkhäuser, Basel, 191-201, 2014.

[2] M. Ferreira, M.M. Rodrigues, N. Vieira. *A time-fractional Borel-Pompeiu formula and a related hypercomplex operator calculus*, Complex Analysis and Operator Theory, 13-No.6, 2495-2526, 2019.

[3] N. Vieira, M. Ferreira, M.M. Rodrigues, R.S. Kraußhar. *Hypercomplex operator calculus for the fractional Helmholtz equation*, Mathematical Methods in the Applied Sciences, 47-No.14, 11439-11472, 2024.

AI approaches to Geometry

Geometric transformations play a central role in the architectures and training of modern AI methods. Whilst geometrically-inspired insights often lead to improved AI techniques, the converse application allows AI methods to provide new means of building, approximating, and examining important geometries. Clifford algebras further provide an algebraic framework to study many important geometric transformations, potentially offering novel compact and coordinate-free architecture modifications. This session explores this intersection, bringing together the array of AI techniques most suited to geometric problems and the geometric-objects most amenable to data representation and machine learning.

Session organisers: Edward Hirst (University of London, UK), Yang-Hui He (University of Oxford, UK), Henrique N. Sá Earp (University of Campinas, Brazil).

Norm and isometries in Clifford algebra

Ramon González Calvet
Institut Pere Calders, Spain
rgonzalezcalvet@gmail.com

Isometries are defined as the linear transformations that commute with any power of every element in a Clifford algebra. In this way, isometries preserve the characteristic polynomial of the matrix representation. Since the coefficients of the characteristic polynomial are invariant [1, p. 147] under isometries, a multiplicative norm is defined for every element in a Clifford algebra as the n^{th}-root of the determinant of its faithful matrix representation [2] having dimension $n \times n$. This definition is independent of the dimension of the matrix representation. Then, direct and indirect isometries are identified, respectively, with the automorphisms and antiautomorphisms of a Clifford algebra. According to the Skolem-Noether theorem, direct isometries can be written as similarity transformation of matrices for even Clifford algebras, and indirect isometries can also be written in this way with additional composition with reversion. Involutions of Clifford algebras and several geometric isometries such as rotations, reflections, and duality are reviewed and expressed in the same formal way. The concept of orthogonality between elements in a Clifford algebra is discussed, and the pseudo-Pythagorean theorem is proven within the scope of the previous definitions. A general definition of units in Clifford algebras is given, and their matrix representations are also reviewed. Finally, we outline how to apply these definitions to general associative algebras.

References

[1] Ch. G. Cullen, *Matrices and Linear Transformations*, 2nd Ed. Dover, New York, 1990.

[2] R. González Calvet, *On Matrix Representations of Geometric (Clifford) Algebras*, J. Geom. Sym. Phys., 43, pp. 1-36, 2017.

From Clifford spinors and ADE correspondences to characteristic multivector invariants and machine learning

Pierre-Philippe Dechant
School of Mathematics, University of Leeds, UK
matpd@leeds.ac.uk

Geometric Algebra's uniquely simple reflection formula makes it the ideal framework in which to study reflection groups and their root systems (and associated algebras and geometries). Furthermore, via the Cartan-Dieudonné theorem, more general transformations can be studied in this framework such as orthogonal and conformal transformations, which are of course of wide interest in computer graphics and related fields. I will review how such a framework leads to a novel set of ADE sets and correspondences [4,1]. It also sheds interesting light on the underlying geometry and its geometric invariants: for instance, the geometry of the Coxeter plane and its siblings where a distinguished type of group element (Coxeter element) acts simply as a rotation by $\frac{2\pi m}{h}$ – for integer h (Coxeter number) and associated prime exponents m – in the plane given by some simple bivector. Moreover, recent work has rekindled interest in certain Geometric Algebra invariants of linear transformations called 'characteristic multivectors' using simplicial derivatives [9,8,7].
Combining these two approaches motivates the calculation of the characteristic multivectors for Coxeter elements for different reflection groups, including those of ADE types. Since larger groups such as in dimension 8 can have large numbers of Coxeter elements, this leads to interesting data sets via a computational algebra approach. This algebraic data set can then be explored using machine learning techniques such as Principal Component Analysis and neural network ternary ADE classification tasks [2].
Time permitting, I will also discuss analogous approaches to combining computational algebra with data science methods in the context of cluster algebras [5,6,3] with similar implications for machine learning algebra and geometry.

References

[1] P. Cameron, P.P. Dechant, Y.H. He, J. McKay, *ADE: patterns in mathematics*, Cambridge University Press, 2025.

[2] S. Chen, P.P. Dechant, Y.H. He, E. Heyes, E. Hirst, D. Riabchenko, *Machine learning Clifford invariants of ADE Coxeter elements*, arXiv:2310.00041, 2023.

[3] M.W. Cheung, P.P. Dechant, Y.H. He, E. Heyes, E. Hirst, J.R. Li, *Clustering cluster algebras with clusters*, Adv. Theor. Math. Phys. 27, 797–828, 2023.

[4] P.P. Dechant, *From the trinity* (A_3, B_3, H_3) *to an ADE correspondence*, Proceedings of the Royal Society A 474(2220), 20180, 034, 2018.

[5] P.P. Dechant, *Applying a computational algebra and data science pipeline to Clifford algebra and cluster algebra data*, Bulgarian Journal of Physics 52(s1), 72–81, 2025.

[6] P.P. Dechant, Y.H. He, E. Heyes, E. Hirst, *Cluster algebras: Network science and machine learning*, Journal of Computational Algebra 8, 100, 008, 2023.

[7] D. Hestenes, S. Sobczyk, *Clifford algebra to geometric calculus: a unified language for mathematics and physics*, vol. 5. Springer Science & Business Media, 2012.

[8] A. Lasenby, *Some recent results for* $SU(3)$ *and octonions within the Geometric Algebra approach to the fundamental forces of nature*, Mathematical Methods in the Applied Sciences, 2022.

[9] A. Lasenby, J. Lasenby, C. Matsantonis, *Reconstructing a rotor from initial and final frames using characteristic multivectors: With applications in orthogonal transformations*, Mathematical Methods in the Applied Sciences, 2022.

Orthogonal equivariant neural networks via generalized Clifford groups

Ekaterina Filimoshina, Dmitry Shirokov
HSE University, Moscow, Russia
efilimoshina@hse.ru

We introduce a novel framework of neural networks equivariant w.r.t. the pseudo-orthogonal (or complex orthogonal) group $\mathrm{O}(V,\mathfrak{q})$ of transformations (including rotations, reflections, etc.) of any n-dimensional vector space V (real $\mathbb{R}^{p,q}$, $n = p+q$, or complex $\mathbb{C}^n$) equipped with a quadratic form $\mathfrak{q}$:

$$\mathrm{O}(V,\mathfrak{q}) := \{P : V \to V, \quad P \text{ is linear, invertible}, \quad P(\mathfrak{q}(v)) = \mathfrak{q}(v), \quad \forall v \in V\}. \qquad (1)$$

A neural network (mapping) $L : X \to Y$ is called $\mathrm{O}(V,\mathfrak{q})$-equivariant [1] if $L(P(x)) = P(L(x))$ for any $P \in \mathrm{O}(V,\mathfrak{q})$ and $x \in X$. This means that we may either first apply any orthogonal transformation to the input data and then apply the network (left-hand side), or first apply the network and then apply the same orthogonal transformation to the output (right-hand side), and the result is guaranteed to be the same. $\mathrm{O}(V, \mathfrak{q})$-equivariant neural networks have various applications in natural science and computer vision, where tasks inherently involve symmetries [2].

Our main idea is that to enforce equivariance of a mapping (layer) w.r.t. $\mathrm{O}(V, \mathfrak{q})$, we can enforce equivariance w.r.t. some larger groups; then the mapping is automatically $\mathrm{O}(V, \mathfrak{q})$-equivariant. It is convenient and natural to introduce these larger groups in terms of geometric (Clifford) algebra $Cl := Cl(V)$ as groups containing the well-known Clifford and Lipschitz groups as subgroups, which perform the same action as $\mathrm{O}(V, \mathfrak{q})$ on V via the adjoint and twisted adjoint actions respectively [3]. The Clifford Γ and Lipschitz $\Gamma^{\pm}$ groups are the groups preserving the grade-1 subspace $V \equiv Cl^1$ under the adjoint and twisted adjoint actions respectively [4]:

$$\Gamma := \{T \in Cl^{\times} : \quad TVT^{-1} \subseteq V\}, \quad \Gamma^{\pm} := \{T \in Cl^{\times} : \quad \widehat{T}VT^{-1} \subseteq V\}, \tag{2}$$

where $Cl^{\times}$ is the group of all invertible elements of Cl and $\widehat{T}$ is the grade involute of T. It is natural to define the groups containing Γ and $\Gamma^{\pm}$ as subgroups by enlarging the subspace Cl^1 in the defining conditions. Therefore, we introduce and study the following classes of 'larger' groups in Cl:

$$\mathrm{P} := \{T \in Cl^{\times} : \quad TCl^{(1)}T^{-1} \subseteq Cl^{(1)}\}, \quad \check{\mathrm{P}} := \{T \in Cl^{\times} : \quad \widehat{T}Cl^{(1)}T^{-1} \subseteq Cl^{(1)}\} \tag{3}$$
$$\mathrm{A} := \{T \in Cl^{\times} : \quad TCl^{\overline{01}}T^{-1} \subseteq Cl^{\overline{01}}\}, \quad \check{\mathrm{A}} := \{T \in Cl^{\times} : \quad \widehat{T}Cl^{\overline{12}}T^{-1} \subseteq Cl^{\overline{12}}\}, \tag{4}$$
$$\mathrm{B} := \{T \in Cl^{\times} : \quad TCl^{\overline{12}}T^{-1} \subseteq Cl^{\overline{12}}\}, \quad \check{\mathrm{B}} := \{T \in Cl^{\times} : \quad \widehat{T}Cl^{\overline{01}}T^{-1} \subseteq Cl^{\overline{01}}\}, \tag{5}$$
$$\mathrm{Q} := \{T \in Cl^{\times} : \quad TCl^{\overline{1}}T^{-1} \subseteq Cl^{\overline{1}}\}, \quad \check{\mathrm{Q}} := \{T \in Cl^{\times} : \quad \widehat{T}Cl^{\overline{1}}T^{-1} \subseteq Cl^{\overline{1}}\}, \tag{6}$$

where Cl^i, $i = 0, \ldots, n$, are the grade subspaces of Cl; $Cl^{(l)} = \bigoplus_{i=l \bmod 2} Cl^i = \{T \in Cl : \quad \widehat{T} = (-1)^l T\}$, $l = 0, 1$, are even and odd subspaces determined by the grade involution $\widehat{\ }$; $Cl^{\overline{k}} = \bigoplus_{i\, =\, k \bmod 4} Cl^i = \{T \in Cl : \quad \widehat{T} = (-1)^k T, \quad \widetilde{T} = (-1)^{\frac{k(k-1)}{2}} T\}$, $k = 0, 1, 2, 3$, are the subspaces determined by the grade involution $\widehat{\ }$ and reversion $\widetilde{\ }$, and $Cl^{\overline{km}} := Cl^{\overline{k}} \oplus Cl^{\overline{m}}$. We prove that the groups (3)–(6) contain the Clifford Γ and Lipschitz $\Gamma^{\pm}$ groups as subgroups:

$$\Gamma \subseteq \mathrm{P},\ \mathrm{A},\ \mathrm{B},\ \mathrm{Q}, \qquad \Gamma^{\pm} \subseteq \check{\mathrm{P}},\ \check{\mathrm{A}},\ \check{\mathrm{B}},\ \check{\mathrm{Q}}, \tag{7}$$

and have equivalent definitions that use only the center of Cl and norm functions applied in the theory of spin groups. Mappings equivariant w.r.t. P, A, B, Q, $\check{\mathrm{P}}$, $\check{\mathrm{A}}$, $\check{\mathrm{B}}$, $\check{\mathrm{Q}}$ are automatically $\mathrm{O}(V, \mathfrak{q})$-equivariant.

We prove that the mappings listed in Table 1 are equivariant w.r.t. the groups P, A, B, Q, $\check{\mathrm{P}}$, $\check{\mathrm{A}}$, $\check{\mathrm{B}}$, $\check{\mathrm{Q}}$. We construct linear, geometric product, and normalization layers by parameterizing with real parameters the linear combinations, geometric products, and norm functions of projections of input multivectors on different subspaces: $Cl^{(l)}$, $l = 0, 1$, $Cl^{\overline{k}}$, $Cl^{\overline{km}}$, $k, m = 0, 1, 2, 3$. We construct and implement various neural network configurations by combining the layers mentioned above. We empirically study the expressiveness–efficiency trade-off induced by different configurations: the larger the group of equivariance, the fewer parameters are needed to parameterize its layers, but the less expressive these layers become. The suggested framework generalizes CGENN proposed in [5].

Table 1: The first column specifies the mapping, the second lists the groups w.r.t. which it is equivariant.

Mappings	Groups
Polynomials $F \in \mathbb{F}[U_1, \ldots, U_l]$	$Cl^{\times}$, P, A, B, Q, Γ, $\check{\mathrm{P}}$, $\check{\mathrm{A}}$, $\check{\mathrm{B}}$, $\check{\mathrm{Q}}$, $\Gamma^{\pm}$
Projections onto $Cl^{\overline{k}}$, $k = 0, 1, 2, 3$	Q, Γ, $\check{\mathrm{Q}}$, $\Gamma^{\pm}$
Projections onto $Cl^{(0)}$ and $Cl^{(1)}$; grade involution $\widehat{\ }$; function $\widehat{T}T$	P, Q, Γ, $\check{P}$, $\check{\mathrm{Q}}$, $\Gamma^{\pm}$
Projections onto $Cl^{\overline{01}}$ and $Cl^{\overline{23}}$; reversion $\widetilde{\ }$; function $\widetilde{T}T$	A, Q, Γ, $\check{\mathrm{A}}$, $\check{\mathrm{Q}}$, $\Gamma^{\pm}$
Projections onto $Cl^{\overline{12}}$ and $Cl^{\overline{03}}$; Clifford conjugation $\widehat{\widetilde{\ }}$; function $\widehat{\widetilde{T}}T$	B, Q, Γ, $\check{\mathrm{B}}$, $\check{\mathrm{Q}}$, $\Gamma^{\pm}$

From a geometric perspective, our work shows how Clifford algebras can be used not only to represent geometric transformations but also to discover new symmetries that simplify learning on geometric data. We illustrate the approach on tasks where geometry is central: convex hull volume estimation, N-body dynamics prediction, prediction of atomic positions in molecular geometries (MD17), and on Lorentz-equivariant problems from high-energy physics (jet tagging).
This talk continues the study started by the authors in the papers [6]-[11] and presented at the conferences ICML 2025 and IJCNN 2025 (Section on Complex- and Hypercomplex-valued Neural Networks).

References

[1] T. Cohen, M. Welling, *Group equivariant convolutional networks.* In Proceedings of the 33rd International Conference on Machine Learning (ICML 2016), volume 48 of Proceedings of MachineLearning Research, 2016.

[2] M. Bronstein, J. Bruna, T. Cohen, P. Veličković, *Geometric deep learning: Grids, groups,graphs, geodesics, and gauges*, arXiv:2104.13478, 2021.

[3] I. Benn, R. Tucker, *An Introduction to Spinors and Geometry with Applications in Physics*, Adam Hilger, Bristol, 1987.

[4] P. Lounesto, *Clifford Algebras and Spinors*, Cambridge University Press, 1997.

[5] D. Ruhe, J. Brandstetter, P. Forré, *Clifford Group Equivariant Neural Networks.* In 37th Conference on Neural Information Processing Systems (NeurIPS 2023), 2023.

[6] E. Filimoshina, D. Shirokov, *GLGENN: A Novel Parameter-Light Equivariant Neural Networks Architecture Based on Clifford Geometric Algebras.* In Proceedings of the 42nd International Conference on Machine Learning (ICML 2025, Vancouver, Canada, 2025), Proceedings of Machine Learning Research, 267, 17153-17188, 2025.

[7] E. Filimoshina, D. Shirokov, *Equivariant Neural Networks with Geometric Algebras: A New Approach*, IEEEProceedings, 1–8, 2025.

[8] E. Filimoshina, D. Shirokov, *On Lie Groups Preserving Subspaces of Degenerate Clifford Algebras*, Advances in Applied Clifford Algebras, 36, 16, 2026.

[9] E. Filimoshina, D. Shirokov, *Generalized Degenerate Clifford and Lipschitz Groups in Geometric Algebras*, Advances in Applied Clifford Algebras, 35, 29, 2025.

[10] E. Filimoshina, D. Shirokov, *On Some Lie Groups in Degenerate Clifford Geometric Algebras*, Advances in Applied Clifford Algebras, 33, 44, 2023.

[11] E. Filimoshina, D. Shirokov, *On generalization of Lipschitz groups and spin groups*, Mathematical Methods in the Applied Sciences, 47:3, 1375–1400, 2024.

GNNs for brane tilings

Tancredi Schettini Gherardini, work in collaboration with Pietro Capuozzo and Benjamin Suzzoni
University of Bonn, Department of Mathematics, Germany
tsg@math.uni-bonn.de

In this talk, a framework for building validated datasets of brane tilings on T^2 and their associated $4d$ $N = 1$ toric quiver gauge theories is presented. It contains physical and geometric information concerning invariants and Seiberg duality orbits. We mainly focus on the $L^{a,b,c}$ family, on which we train Graph Neural Networks (GNNs) that regress toric invariants from the dimer graph and surface patterns within duality orbits (learnt as a local graph rewrite).

Versor: A geometric sequence architecture enhanced scale generalization and interpretability via conformal algebra

Truong Minh Huy, Edward Hirst
Independent Researcher, Vietnam
louistruong1111@gmail.com

A novel sequence architecture is introduced, Versor, which uses Conformal Geometric Algebra (CGA) in place of traditional linear operations to achieve structural generalization and significant performance improvements on physical systems, while offering improved interpretability and efficiency. By embedding states in the $Cl_{4,1}$ manifold and evolving them

via geometric transformations (rotors), Versor natively represents $SE(3)$-equivariant relationships and structurally mirrors physical conservation laws through pseudo-orthogonal transformations in Clifford representation space.
Versor is validated on chaotic N-body dynamics, topological reasoning, and complex physical regimes, demonstrating exceptional out-of-distribution (OOD) generalization by achieving a 93.0% performance improvement in unconstrained open-space escape scenarios with heavier masses, where standard Transformers degrade by +775.9%. Key results include: orders-of-magnitude fewer parameters (370× vs. Transformers); interpretable attention decomposing into proximity and orientational components; zero-shot scale generalization (0.993 vs. 0.070 MCC for ViT, alongside $> 99.6\%$ accuracy on a balanced dataset); and a Recursive Rotor Accumulator (RRA) providing $O(L)$ linear temporal complexity. We mathematically prove that rotor evolution in $\mathcal{C}l_{4,1}$ acts via pseudo-orthogonal transformations in the Clifford representation space, yielding strong geometric inductive biases that stabilize physical rollouts. Custom Clifford kernels achieve a cumulative over 100× speedup over naive geometric algebra implementations, reducing total per-rollout latency to 0.88 ms with manageable constant-factor overhead relative to highly-optimized linear Transformers.

References

[1] J. Brehmer, P. De Haan, S. Behrmann, T. Cohen, *Geometric algebra transformer*, Advances in Neural Information Processing Systems, volume 36, 2023.

[2] M.M. Bronstein, J. Bruna, T. Cohen, P. Velickovic, *Geometric deep learning: Grids, groups, graphs, geodesics, and gauges*, arXiv:2104.13478, 2021.

[3] S. Chen, P.-P. Dechant, Y.-H. He, E. Heyes, E. Hirst, D. Riabchenko, *Machine Learning Clifford Invariants of ADE Coxeter Elements*, Adv. Appl. Clifford Algebras, 34(3):20, 2024.

[4] S. Chmiela, A. Tkatchenko, H. E. Sauceda, I. Poltavsky, K. T. Schütt, K.-R. Müller, *Machine learning of accurate energy-conserving molecular force fields*, Science advances, 3(5):1603015, 2017.

[5] J. Cohen, *Statistical Power Analysis for the Behavioral Sciences.* Lawrence Erlbaum Associates, 2nd edition, 1988.

[6] T. Cohen, M. Welling, *Group equivariant convolutional networks* In: International Conference on Machine Learning, pp. 2990-2999, PMLR, 2016.

[7] NVIDIA Corporation, *CUDA C++ Programming Guide*, Version 12.3, 2023.

[8] C. Doran, A. Lasenby, *Geometric algebra for physicists.* Cambridge University Press, 2003.

[9] L. Dorst, D. Fontijne, S. Mann, *Geometric algebra for computer science.* Morgan Kaufmann, 2007.

[10] A. Dosovitskiy et al., *An image is worth 16x16 words: Transformers for image recognition at scale.* In: International Conference on Learning Representations, 2021.

[11] S. Greydanus, M. Dzamba, J. Yosinski, *Hamiltonian neural networks*, Advances in Neural Information Processing Systems, volume 32, 2019.

[12] A. Gu, T. Dao, *Mamba: Linear-time sequence modeling with selective state spaces*, arXiv:2312.00752, 2023.

[13] K. He, X. Zhang, S. Ren, J. Sun, *Delving deep into rectifiers: Surpassing human-level performance on imagenet classification.* In: Proceedings of the IEEE International Conference on Computer Vision (ICCV), pp. 1026-1034, 2015.

[14] E. Hirst, T. Schettini Gherardini, A. G. Stapleton, *AInstein: Numerical Einstein Metrics via Machine Learning*, Machine Learning: Science and Technology, 2025.

[15] J. Jumper, R. Evans, A. Pritzel, et al., *Highly accurate protein structure prediction with alphafold*, Nature, 596(7873):583-589, 2021.

[16] A. Krizhevsky, G. Hinton, et al., *Learning multiple layers of features from tiny images*, Technical report, University of Toronto, 2009.

[17] D. Linsley, J. Kim, V. Veerappan, C. Windolf, T. Serre, *Learning long-range spatial dependencies with horizontal gated recurrent units*, Advances in Neural Information Processing Systems (NeurIPS), 31, 2018.

[18] I. Loshchilov, F. Hutter, *Decoupled weight decay regularization.* In: International Conference on Learning Representations, 2019.

[19] S. Merity, C. Xiong, J. Bradbury, R. Socher, *Pointer sentinel mixture models*, arXiv:1609.07843, 2016.

[20] M. Raissi, P. Perdikaris, G. E. Karniadakis, *Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations*, Journal of Computational Physics, 378:686-707, 2019.

[21] D. Ruhe et al., *Clifford group equivariant neural networks*, Advances in Neural Information Processing Systems, volume 36, 2023.

[22] A. Sanchez-Gonzalez, J. Godwin, T. Pfaff, R. Ying, J. Leskovec, P. Battaglia, *Learning to simulate complex physics with graph networks.* In: International Conference on Machine Learning, pp. 8459-8468. PMLR, 2020.

[23] V. G. Satorras, E. Hoogeboom Ee, M. Welling, *$E(n)$ equivariant graph neural networks.* In: International conference on machine learning, pp. 9323-9332. PMLR, 2021.

[24] S. Sriram, T. Lüddecke, A. Paliwal, A. S. Ecker, C. van de Geijn, *Clifford algebraic rotor embeddings: Maybe embeddings should start to care*, arXiv:2511.11665, 2025.

[25] The PyGAE Team, *Clifford: Geometric algebra for python*, 2025.

[26] P. Tillet, H.-T. Kung, D. Cox, *Triton: an intermediate language and compiler for tiled neural network computations.* In: Proceedings of the 3rd ACM SIGPLAN International Workshop on Machine Learning and Programming Languages, pp. 10-19, 2019.

[27] A. Vaswani et al., *Attention is all you need*, Advances in Neural Information Processing Systems, volume 30, 2017.

[28] S. Williams, A. Waterman, D. Patterson, *Roofline: an insightful visual performance model for multicore architectures*, Communications of the ACM, 52(4):65-76, 2009.

[29] M. Wortsman, G. Ilharco, S. Y. Gadre, R. Roelofs, R. Gontijo-Lopes, A. S. Morcos, H. Namkoong, A. Farhadi, Y. Carmon, S. Kornblith, et al., *Model soups: averaging weights of multiple fine-tuned models improves accuracy without increasing inference time.* In: International Conference on Machine Learning, pp. 23965-23998. PMLR, 2022.

Null geometric algebra: An ideal tool for AI-aided geometric reasoning

Hongbo Li
Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, China
hli@mmrc.iss.ac.cn

Automated Theorem Proving started at the very beginning of artificial intelligence. In 1956, Newell and Simon wrote a mathematical theorem proving program named "LOGIC THEORIST" [1], and initiated the first step of logical reasoning by computers. This program can be used to prove some theorems in Bertrand Russell's "The Principles of Mathematics". In 1958, Hao Wang of Rockefeller University designed a program called "Wang's algorithm" [2], which can prove all the theorems in "The Principles of Mathematics" (more than 350 theorems). In 1976, Appel and Haken proved the four color theorem by logic deduction with the help of computers. In 1977, Wen-Tsun Wu of the

Institute of Systems Science, Chinese Academy of Sciences, proposed "Wu method" [3], which used computer algebra to prove elementary geometry theorems efficiently. In the 1980's, the Gröbner basis method, proposed by Buchberger, occurred as another powerful algebraic method for automated theorem proving in elementary geometry. In the last 50 years, the logic approach and the algebraic approach are the two most popular and at the same time mutually competitive approaches to Automated Theorem Proving.

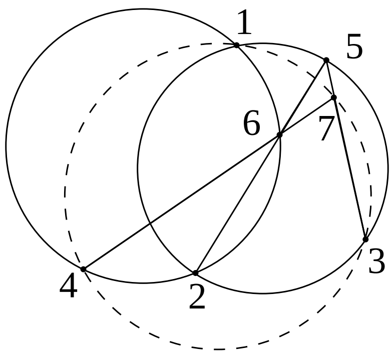
With the rise of deep learning and LLM in recent years, AI becomes an important tool not only for mathematical education and competition, but also for mathematical research. Indeed, the last two months have witnessed a wave of mathematical problems at the forefront of pure and applied mathematics being solved by AI or with the aid of AI. At the contest level, AlphaGeometry is a mathematical reasoning system developed by DeepMind that specifically targets at complicated geometric problem solving. AlphaGeometry 2, by integrating Gemini language model with symbolic engine, successfully defeats average human IMO Gold Medalists with a rate of success 84% in tackling the IMO competition problems, far exceeding Wu method. Still this system is liable to mathematical mistakes because its reasoning is not based on formal logic. At the research level, when restricted to the mathematical research teams in China, based on the formal logic language LEAN and the Matlas natural language semantics search system, the AI4Math team of Peking University successfully proved the Anderson Conjecture in commutative algebra, and the MechMath-Agent [4] team of AMSS, Chinese Academy of Sciences, successfully solved four open problems on sparse polynomial computation and one open problem on differential Galois group, demonstrating the power of AI-aided research in the front of mathematics.

When it comes to Geometric Algebra, a version of Clifford algebra with emphasis on coordinate-free representation and manipulation of geometric relations on one hand, and geometric interpretations of algebraic expressions on the other hand, in the 1990's Hongbo Li applied Geometric Algebra to automated theorem proving of classical geometry, differential geometry, hyperbolic geometry, etc. [5], and developed efficient methods based on vectorial equation-solving and Wu's characteristic set method. In the 2000's, Hongbo Li proposed a method based on Null Geometric Algebra [6], a version of Conformal Geometric Algebra where all the atomic non-scalar generators are null vectors; the method can generate extremely short proofs for many theorems in classical geometry: the proofs are mostly consisting of one or two terms at each step. In the 2010's, Hongbo Li further extended the method to a method of automated quantitizing of geometric theorems [7], which can be used to make automated extension of classical theorems by removing one or more geometric constraints of equality type in the geometric configuration, and finding the quantitative geometric relationship between the missing constraints and the conclusion expression.

The Geometric Algebra approach to Automated Theorem Proving in classical geometry has the advantage of nice geometric interpretations for its sufficiently short algebraic expressions, and hence is beneficial for extending geometric theorems beyond their original configurations, and is more suitable for current LLM-based geometric reasoning. This talk first showcases the power of Null Geometric Algebra in efficiently extending classical geometric theorems by reducing the number of constraints in the geometric configurations, then discloses the intrinsic properties of Null Geometric Algebra supporting the efficiency, from both the algebraic and the geometric perspectives [8].

In the following we present a simple but typical example of automated extension of geometric theorem with Null Geometric Algebra, where at each step the number of terms of the conclusion expression remains to be one, hence the geometric deduction is the shortest analytically.

Illustrative Example. Let $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ be free points, and let point $\mathbf{5}$ be on circle $\mathbf{123}$. Let $\mathbf{6}$ be the second intersection of circle $\mathbf{124}$ and line $\mathbf{25}$, and let $\mathbf{7} = \mathbf{35} \cap \mathbf{46}$. Then $\mathbf{1}, \mathbf{3}, \mathbf{4}, \mathbf{7}$ are co-circular.



We remove the co-circularity of points $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{5}$, i.e., remove $[\mathbf{1235}] = 0$. The new configuration can be constructed as follows:

Free points: $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$.

Intersections (second intersection of two circles/lines): $\mathbf{6} = \mathbf{214} \cap \mathbf{2}\mathbf{e}_\infty\mathbf{5}$, $\quad \mathbf{7} = \mathbf{e}_\infty\mathbf{35} \cap \mathbf{e}_\infty\mathbf{46}$.

Conclusion: $[\mathbf{1347}] = 0$.

Now the conclusion expression no longer equals zero in the new configuration, so we need to measure how far the four points $\mathbf{1}, \mathbf{3}, \mathbf{4}, \mathbf{7}$ are away from being co-circular. We compute the expression $[\mathbf{1347}]$ by eliminating the two points of intersection $\mathbf{6}, \mathbf{7}$ in Null Geometric Algebra: let $\mathbf{x} = (\mathbf{1} \wedge \mathbf{4}) \vee_{\mathbf{2}} (\mathbf{e}_\infty \wedge \mathbf{5})$, $\mathbf{y} = (\mathbf{3} \wedge \mathbf{5}) \vee_{\mathbf{e}_\infty} (\mathbf{4} \wedge \mathbf{6})$, then $\mathbf{6} = \mathbf{x2x2}$, and $\mathbf{7} = \mathbf{y}\mathbf{e}_\infty\mathbf{y2}$. So

$$
\begin{aligned}
[\mathbf{1347}] &\overset{\mathbf{7}}{=} -2^{-1}[\mathbf{314}\{(\mathbf{3} \wedge \mathbf{5}) \vee_{\mathbf{e}_\infty} (\mathbf{4} \wedge \mathbf{6})\}\mathbf{e}_\infty\{(\mathbf{3} \wedge \mathbf{5}) \vee_{\mathbf{e}_\infty} (\mathbf{4} \wedge \mathbf{6})\}] \\
&\overset{expand}{=} \underbrace{-2^{-1}[\mathbf{e}_\infty\mathbf{345}][\mathbf{e}_\infty\mathbf{346}]}[\mathbf{3146}\mathbf{e}_\infty\mathbf{5}] \\
&\overset{\mathbf{6}}{=} 2^{-1}[\mathbf{314}\{(\mathbf{1} \wedge \mathbf{4}) \vee_{\mathbf{2}} (\mathbf{e}_\infty \wedge \mathbf{5})\}\mathbf{2}\{(\mathbf{1} \wedge \mathbf{4}) \vee_{\mathbf{2}} (\mathbf{e}_\infty \wedge \mathbf{5})\}\mathbf{e}_\infty\mathbf{5}] \\
&\overset{expand}{=} \underbrace{2^{-1}[\mathbf{e}_\infty\mathbf{124}][\mathbf{e}_\infty\mathbf{245}]}[\mathbf{314125}\mathbf{e}_\infty\mathbf{5}] \\
&\overset{simplify}{=} \underbrace{2^2(\mathbf{e}_\infty \cdot \mathbf{5})(\mathbf{1} \cdot \mathbf{4})}\,[\mathbf{1235}].
\end{aligned}
$$

In the result, that the red-colored bracket being equal to zero is exactly the missing constraint, while all other underlined factors are irrelevant to the missing constraint. Recollecting the factors, we get the following extension of the original theorem: *For any free points* $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$ *in the plane and intersections:* $\mathbf{6} = \mathbf{214} \cap \mathbf{2e_\infty 5}$, $\mathbf{7} = \mathbf{e_\infty 35} \cap \mathbf{e_\infty 46}$,

$$\frac{[\mathbf{1347}]}{\mathbf{3} \cdot \mathbf{7}} = -2\frac{\mathbf{1} \cdot \mathbf{4}[\mathbf{e_\infty 345}][\mathbf{1235}]}{\mathbf{3} \cdot \mathbf{5}[\mathbf{e_\infty 34125}]},$$

where every factor has clear geometric meaning.

References

[1] A. Newell, H.A. Simon, *The logic theory machine: A complex information processing system*, IRE Transactions on Information Theory, 2(3): 61-79, 1956.

[2] H. Wang, *Toward mechanical mathematics*, IBM Journal of Research & Development, 4(1): 2-22, 2010.

[3] W.-T. Wu, *Decision problem in elementary geometry and mechanical proving*, Science in China Mathematics (in Chinese), 20(6): 507-516, 1977.

[4] MechMath official website: https://mechmath.github.io/

[5] H. Li, *Some applications of Clifford to geometries.* In: Gao, X. et al. (eds), Automated Deduction in Geometry, LNAI 1669, Springer, Berlin, Heidelberg, pp. 156-179, 1998.

[6] H. Li, *Invariant Algebras and Geometric Reasoning*, World Scientific, Singapore, 2008.

[7] H. Li, *Automated Geometric Reasoning with Geometric Algebra: Practice and Theory.* In: Proc ISSAC, 2017.

[8] H. Li, *Null Geometric Algebra with Geometric Interpretation*, Philosophical Transactions A. Theme Issue: Modern Applications of Geometric Algebra, E. Hitzer et al. (eds), 2026.

Neural and numerical methods for G_2-structures on contact Calabi-Yau 7-manifolds

Elli Heyes, Edward Hirst, Henrique N. Sá Earp, Tomás S. R. Silva
University of Campinas - UNICAMP, Brazil
tomas@ime.unicamp.br

A numerical framework for approximating G_2-structure 3-forms on contact Calabi-Yau manifolds is presented. The approach proceeds in three stages: first, existing neural network models are employed to compute an approximate Ricci-flat metric on a Calabi-Yau threefold. Second, using this metric and the explicit construction of a G_2-structure on the associated 7-dimensional Calabi-Yau link in the 9-sphere, numerical approximations of the 3-form are generated on a large set of sampled points. Finally, a dedicated neural architecture is trained to learn the 3-form and its induced Riemannian metric directly from data, validating the learned structure and its torsion via a numerical implementation of the exterior derivative, which may be of independent interest.

References

[1] E. Heyes, E. Hirst, H. N. Sá Earp, T. S. R. Silva, *Neural and numerical methods for* G_2*-structures on contact Calabi-Yau 7-manifolds*, Physics Letters B, Vol. 878, 2026.

Clifford algebra neural surrogates for science and engineering

Julian Suk, Patryk Rygiel
Technical University of Munich, Department of Computer Science, Germany
julian.suk@tum.de

Neural surrogate modelling has been gaining traction in science and engineering. The underlying idea is to augment or replace time-intensive numerical simulations by neural networks trained end-to-end on measurements or simulated data. This is commonly done in the form of neural operators [2] that are implemented as transformers [9]. Many engineering and physics problems are subject to the Euclidean symmetries comprising the symmetry group E(3). Geometric algebra transformers [3] (GATr) are powerful equivariant models respecting these symmetries.
GATr has been extended to fit various applications. These range from biomedicine [8] to high-energy physics [6], multiscale modelling [7] and fluid flow [5]. These application mainly benefit from its equivariance achieved through defining hidden states and model

layers via the Clifford (or "geometric") algebras $\mathbb{G}_{3,0,1}$ and $\mathbb{G}_{1,3}$.
Besides equivariant models, the anchored-branched universal physics transformer [1] (AB-UPT) has recently been proposed as neural operator transformer that is feasible on operational scale in automotive aerodynamics. We propose a Clifford algebra variant, AB-GATr [5] and show how its E(3) equivariance is beneficial in cardiovascular hemodynamics modelling *in contrast* to automotive aerodynamics.
We relate this phenomenon to the concept of distributional symmetry breaking [4] which exposes canonical alignment inherent in the training data in the context of neural network training.

References

[1] B. Alkin, M. Bleeker, R. Kurle, T. Kronlachner, R. Sonnleitner, M. Dorfer, J. Brandstetter, *AB-UPT: Scaling neuralCFD surrogates for high- fidelity automotive aerodynamics simulations via anchored branched universal physics transformers*, Transactions on Machine Learning Research,2025.

[2] K. Azizzadenesheli, N. Kovachki, Z. Li, M. Liu-Schiaffini, J. Kossaifi,A. Anandkumar, *Neural operators for accelerating scientific simulations and design*, Nature Reviews Physics, 6, 04 2024.

[3] J. Brehmer, P. De Haan, S. Behrends, T. Cohen, *Geometric algebra transformer.* In: Thirty-seventh Conference on Neural Information Processing Systems, 2023.

[4] H. Lawrence, E. Hofgard, V. Portilheiro, Y. Chen, T. Smidt, R. Walters, *To augment or not to augment? Diagnosing distributional symmetry breaking*, In: The Fourteenth International Conference on Learning Representations, 2026.

[5] P. Rygiel, J. Suk, K.K. Yeung, C. Brune, J.M. Wolterink, *Symmetry in the wild: The role of equivariance in neural fluid surrogates*, 2026.

[6] J. Spinner, V.B. Pla, P. De Haan, T. Plehn, J. Thaler, J. Brehmer, *Lorentz-equivariant geometric algebra transformers for high-energy physics.* In: The Thirty-eighth Annual Conference on Neural Information Processing Systems, 2024.

[7] J. Suk, G. Cesa, T. Hehn, A. Behboodi, *ViNE-GATr: scaling geometric algebra transformers with virtual nodes embeddings.* In: ICLR 2025Workshop on Machine Learning Multiscale Processes, 2025.

[8] J. Suk, B. Imre, J.M. Wolterink, *LaB-GATr: Geometric algebra transformers for large biomedical surface and volume meshes*, Medical Image Computing and Computer Assisted Intervention – MICCAI 2024, pp. 185–195, Cham,2024. Springer Nature Switzerland.

[9] A. Vaswani, N. Shazeer, N. Parmar, J. Uszkoreit, L. Jones, A.N. Gomez, Ł. Kaiser, I. Polosukhin, *Attention is all you need*, Advances in Neural Information Processing Systems, volume 30. Curran Associates, Inc., 2017.

Clifford algebra and the fundamental forces of nature (CAFFN)

Of interest for this session are applications of Clifford algebra, geometric calculus, and octonions to the physics of strong, electromagnetic, weak, and gravitational fields. Topics could include geometric and Clifford algebra approaches to understanding fundamental physical interactions, and current research connecting octonions to Standard Model symmetries and beyond. Applications to topics within gravitational theory such as dark matter and dark energy are also welcome.

Session organisers: Gene McClellan (Applied Research Associates, USA), Anthony Lasenby (Cambridge University, UK).

An algebraic solution of the non-Hermitian Liouville-von Neumann equation for two-state systems in Clifford algebra

Pedro Amao, Hernan Castillo
Pontifical Catholic University of Peru, Peru
pedro.amao@pucp.edu.pe

In this work, we investigate a two-state system coupled to a generic dissipative environment. Conventionally, such systems are described using a non-Hermitian Hamiltonian within the complexified Pauli matrix algebra, and their dynamics are governed by the non-Hermitian Liouville-von Neumann equation, which is traditionally solved via matrix methods to find the density operator $\hat{\rho}(t)$. By leveraging the isomorphism between the Pauli matrix algebra $\mathrm{Mat}(2, \mathbb{C})$ and the Clifford algebra of three-dimensional space Cl_3, we completely translate the non-Hermitian problem into the geometric framework of Cl_3. This approach natively decouples the dynamics into distinct geometric grades, successfully bypassing the need for cumbersome matrix techniques, such as finding the eigenvalues and eigenvectors of a 4×4 matrix. We demonstrate that the non-unitary time evolution is driven by a generalized geometric spinor that seamlessly unifies the dissipative decay and the Hamiltonian rotation. Finally, we test our general algebraic solution in the Hermitian regime, showing perfect agreement with well-known unitary results.

References

[1] A. Sergi, K.G. Zloshchastiev, *Non-Hermitian quantum dynamics of a two-level system and models of dissipative environments*, International Journal of Modern Physics B, 27:27, 2013.

[2] P. Amao, H. Castillo, *The Clifford Algebra of the Density Matrix: An Elementary Approach*, Advances in Applied Clifford Algebras, 34, 2024.

Standard Model Symmetries and the Unsung Algebra $\mathbb{O} \oplus \mathbb{H} \oplus \mathbb{C} \oplus \mathbb{R}$

Nichol Furey
Universität zu Berlin, Institute of Physics, Germany
furey@physik.hu-berlin.de

Where do the Standard Model's internal symmetries come from? Treating $\mathbb{O} \oplus \mathbb{H} \oplus \mathbb{C} \oplus \mathbb{R}$ as a module for its own multiplication algebra enables a particular origin story for the Standard Model's pre-Higgs, $\mathfrak{g}_{\text{SM}} := \mathfrak{su}(3)_C \oplus \mathfrak{su}(2)_L \oplus \mathfrak{u}(1)_Y$, and post-Higgs, $\mathfrak{g}_{\text{LE}} := \mathfrak{su}(3)_C \oplus \mathfrak{u}(1)_Q$, symmetries. Here we find that weak hypercharge and electric charge operators, Y and Q, take on a remarkably simple form: $\sum \frac{1}{n}\mathbb{I}_{n\times n}$.
With the help of Cayley-Dickson imaginary units, this $15\,\mathbb{R}$ dimensional $\mathbb{O} \oplus \mathbb{H} \oplus \mathbb{C} \oplus \mathbb{R}$ embeds naturally as a vector space into several well-studied $16\,\mathbb{R}$ dimensional algebras, which we generically refer to as $\mathbb{V}$. With this embedding, the Standard Model's internal symmetries may then be seen to arise in part from the sequence of nested inclusions: $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O} \subset \mathbb{V}$. In the sedenionic case of $\mathbb{V} = \mathbb{S}$, the full sequence becomes a Cayley-Dickson tower. We define the notion of *endomorphic models* of particle physics, and connect $\text{End}_{\mathbb{R}}(\mathbb{V}) \simeq Cl(0,8)$ to the earlier ideas of *Bott Periodic Particle Physics*.

References

[1] N. Furey, *Standard Model Symmetries and the Nested Embeddings of* $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$, arXiv:2607.18450.

[2] N. Furey, *A Superalgebra Within: representations of lightest standard model particles form a $\mathbb{Z}_2^5$-graded algebra*, Annalen der Physik, arXiv:2505.07923, 2025.

Møller scattering from relational closure in $\mathrm{Cl}(1,3)$

Colin Hayhurst
Independent researcher, Haywards Heath, UK
mailto:colinhayhurst@fastmail.com

The Spacetime Algebra (STA) formulation of Dirac theory was pioneered by Hestenes [1], who reformulated the S-matrix as a real spinor-valued operator. He demonstrated that cross-sections can be calculated without spin sums at the single-particle level. Lewis, Lasenby and Doran [2,3] extended this to Compton scattering, pair annihilation, and second-order Coulomb calculations. However, they explicitly identified that this method does not carry over neatly to the multiparticle case, saying they "still have to resort to spin sums over terms involving complex conserved currents". They did not address identical particle scattering, where the conventional approach additionally requires Fock-space antisymmetry for the exchange sign, and diagram enumeration for channel topology. Møller scattering, the simplest identical-fermion process, is in this second, unaddressed category. We show that both limitations are overcome with a new approach. In this, particles are represented as intrinsic rotors and interactions are formulated as relational closure conditions on multiparticle loops. The theoretical framework we use is the real Clifford algebra $\mathrm{Cl}(1,3)$, now applied to local frames.

A fundamental object in our approach is the holonomy of a closed loop, constrained to lie in the spin centre of $\mathrm{Spin}(1,3)$. Channel topology and the exchange sign emerge from the grade algebra of chiral multivectors.

The key results are as follows:

(i) Channel Topology Theorem. In our approach each lepton species has a chirality multivector built from the pseudoscalar and angular bivector of the local frame. These are denoted $\iota_\chi^+ = I + e_{\theta\phi}$ for the electron, and $\iota_\chi^- = I - e_{\theta\phi}$ for the positron. We prove that the $t+u$ structure of Møller scattering, with no s-channel contribution, follows as a theorem from the grade algebra of the same-species geometric product of $\iota_\chi^+\iota_\chi^+$, before any cross-section is computed. While the absence of an s-channel is elementary from charge conservation, the broader significance is that the same algebraic fact simultaneously fixes channel topology for all two-body lepton processes. The same theorem yields the $t+s$ structure of Bhabha scattering from the $\iota_\chi^+\iota_\chi^-$ product. Channel topology is thus fixed algebraically by particle species, not by enumerating diagrams.

(ii) Exchange sign from holonomy. We derive the relative minus sign between the t- and u-channel amplitudes from the exchange holonomy in $\mathrm{Spin}(1,3)$. This sign reproduces the Fermi exchange sign without the need for Fock-space antisymmetry as a postulate.

(iii) Spin sum eliminated. The polarised amplitude has interference coefficient -2 between the t- and u-channels, fixed by the exchange sign. We obtain the unpolarised Mott coefficient -1 by averaging over the four definite-spin configurations of the two-particle closure condition. Their equal weights are fixed by an automorphism-invariance theorem, using only the grade algebra of $\mathrm{Cl}(1,3)$. No spin-sum completeness relation is used at any stage. The unpolarised coefficient -1 is the standard result, and the per-configuration coefficients $\{-2,-2,0,0\}$ are the expected values. However, both follow directly from closure conditions, without spin sums or completeness relations. This is a direct resolution of the multiparticle spin-sum limitation identified in [2,3].

(iv) Non-relativistic Mott cross-section. We show that the Lorentz-invariant phase-space measure is not an independent postulate but follows from the on-shell closure condition together with Lorentz covariance, and without invoking Fock-space normalisation. The Mott cross-section then follows from the squared grade-4 content of the total transition rotor.

(v) CPT invariance. We show that CPT invariance is a direct algebraic identity in $\mathrm{Cl}(1,3)$ for the abelian electromagnetic two-body lepton processes considered here. The five premises of the Lüders–Pauli–Jost theorem are not required. Our framework has no background Lorentz group, no microcausality of field operators, no Hermitian Hamiltonian, no positive-energy vacuum, and no Hilbert space of states.

The background-free relational network, realised in real $\mathrm{Cl}(1,3)$, requires no renormalisation, virtual particles, or gauge fixing. The results demonstrate that loop holonomy suffices to carry the full identical-fermion scattering problem without using the standard S-matrix approach.

References

[1] D. Hestenes, *Geometry of the Dirac Theory*, Symposium on the Mathematics of Physical Space-Time, Mexico City, 67-96, 1981.

[2] A. Lewis, A.N. Lasenby, C.J.L. Doran, *Electron scattering in the spacetime algebra.* In: R. Ablamowicz and B. Fauser (eds.), Clifford Algebras and Their Applications in Mathematical Physics, Vol. 1, pp. 49-71. Birkhäuser, Boston, 2000.

[3] A. Lewis, C.J.L. Doran, A.N. Lasenby, *Electron scattering without spin sums*, Int. J. Theor. Phys., 40(1), 363-376, 2001.

Para-complex numbers and para-quaternions

Eckhard Hitzer, Stephen J. Sangwine
International Christian University, Tokyo, Japan
hitzer@icu.ac.jp

In this contribution we study para-complex numbers and para-quaternions. The two-dimensional flexible, para-unital (non-unital), commutative, non-associative, non-alternative, normed, composition, division algebra of para-complex numbers is related to standard complex numbers by complex conjugation of its factors $p * q = \bar{p}\bar{q}$. The four-dimensional algebra of para-quaternions shares all the properties of para-complex numbers, is however not commutative and is analogously related to W.R. Hamilton's quaternions by quaternion conjugation of its factors $p*q = \bar{p}\bar{q}$. Importantly, para-complex numbers and para-quaternions appear as natural subalgebras of both para-octonions and of the Okubo algebra of quantum chromo dynamics (QCD) of the standard model of elementary particles. We study in both algebras the exponential minus one functions, polar forms, idempotents, and for para-complex numbers the dihedral groups D_3, D_4 and D_{12} that arise from products with units and idempotents. The embeddings in complex numbers and quaternions, respectively, invite to consider para-Okubo algebra (product of Okubo-conjugated Okubo elements).

References

[1] D. Corradetti, A. Marrani, F. Zucconi, *A minimal and non-alternative realisation of the Cayley plane*, Annali Dell Universita Ferrara 70, pp. 681-730, 2024.

[2] L. Dorst, S. De Keninck, *A Guided Tour to the Plane-Based Geometric Algebra PGA (Version 2.0)*. 14 Mar. 2022.

[3] P. Girard, *Quaternions, Clifford Algebras and Relativistic Physics*, Birkhäuser, Basel, 2007.

[4] E. Hitzer, *Creative Peace License.* http://gaupdate.wordpress.com/2011/12/14/the-creative-peace-license-14-dec-2011/, access: 12 June 2020.

[5] E. Hitzer, M. Kamarianakis, G. Papagiannakis, P. Vasik, *Survey of New Applications of Geometric Algebra*, Mathematical Methods in the Applied Sciences, 47(14), pp. 11368-11384, 2024.

[6] E. Hitzer, S. J. Sangwine, *Embedding the Okubo algebra in Clifford's geometric algebras and vice versa*, in preparation.

[7] E. Hitzer, *Contemplating Susumu Okubo's Algebra of Quantum Color Dynamics*, in preparation.

[8] S. Ivanov, M. Tchomakova, S. Zamkovoy, *Geometry of paraquaternionic contact structures*, Differential Geometry and its Applications, Volume 100, 102274, 2025.

[9] S. Kaneyuki, M. Kozai, *Paracomplex Structures and Affine Symmetric Spaces*, Tokyo Journal of Mathematics, Vol. 8, No. 9, pp. 81-98, 1985.

[10] A. Marrani, D. Corradetti, F. Zucconi, *Physics with non-unital algebras? An invitation to the Okubo algebra*, J. Phys. A: Math. Theor., 58(7) 075202, pp. 1-18, 2025.

[11] S. Okubo, *Pseudo-quaternion and pseudo-octonion algebras*, Hadronic J., vol. 1:4, Oct. 1978.

[12] S. Okubo, *Introduction to Octonion and Other Non-Associative Algebras in Physics*, Cambridge University Press, Cambridge, 1995.

[13] S. J. Sangwine, E. Hitzer, *Clifford Multivector Toolbox (for MATLAB)*, AACA, 27(1), pp. 539-558, 2017.

[14] S. J. Sangwine, E. Hitzer, *Hypercomplex Algebra Toolkit (for MATLAB)*, Software repository: https://sourceforge.net/projects/hypercomplex-algebra-toolkit/.

Electroweak internal symmetry, color and generation $\mathbb{Z}_3$ structures in the real Clifford algebra $\mathcal{C}\ell(3,1)$

Nikolay Ketsaris
Independent researcher, Israel
mailto:n.ketsaris@gmail.com

Numerous attempts to describe the full spectrum of fundamental particles of the Standard Model (SM) by means of Clifford algebras typically enlarge the dimension of the dynamical space (the space of wave functions) $\mathbb{S}$ while keeping the kinematic (coordinate) space $\mathbb{X}$ four-dimensional [1,2]; less often, the kinematic space itself is enlarged as well [3,4]. Here we present an approach to incorporating the full SM gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ within the 16-dimensional vector space of the algebra $\mathcal{C}\ell(3,1,\mathbb{R})$, which we regard as a natural generalization of Minkowski spacetime $\mathbb{X}_{3,1}$. In essence, this is a direct continuation of the C-space concept developed by Pavšič [5], Castro Perelman, and others. In our approach the dynamical and kinematic spaces are of the same kind: $SS = \mathbb{X} = \mathcal{C}\ell(3,1,\mathbb{R})$.
The construction proceeds in two stages.
When the wave function is extended to a full multivector of the Clifford algebra $\mathcal{C}\ell(3,1,\mathbb{R})$, the generalized Dirac equation describes a fermion pair split by mass. We first obtain the

electroweak set $\mathfrak{E} = \{\mathcal{A}, \mathcal{B}, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{M}\}$ – the gauge matrices and the mass operator of a fermion pair of one generation. The principal tool for uncovering the electroweak structure is the regular representation of the algebra $\mathcal{C}\ell(3,1)$. We construct two dual sets of 16×16 structure matrices in block form: a left set L^I for the Dirac matrices and a right set R_I for the gauge matrices and the mass operator, as proposed in [6]. The passage from the geometric basis to a "physical" one, in which $L^\mu \sim i\,\mathrm{diag}(\gamma^\mu, \gamma^\mu)$, together with the block structure, allows us to identify all components of the classical Weinberg-Salam equations for a single generation of leptons in an electroweak field.

The resulting gauge matrices of charge, hypercharge, and weak isospin, together with the mass operator, are then expanded in products of the basis elements of the Clifford algebra in its left and right regular representations. The approach carries over naturally to the u and d quarks. We also note that Hestenes's description of the electroweak symmetry within geometric algebra [7], though conceptually close, differs in its substantive results.

In the second stage we generalize the electroweak set $\mathfrak{E}$ to color and generations by using its internal isomorphism with respect to two permutation groups $\mathbb{Z}_3$: a homogeneous bivector one $\{e_{12}, e_{23}, e_{31}\}$ for color, and a grade-inhomogeneous quaternionic one $\{e_{1234}, e_{123}, e_4\}$ for generations. In our scheme the strong interaction and generation mixing arise from a different mechanism—the superposition of modes of a single wave function describing the complete system of fundamental SM particles. We assume that the stable or quasi-stable, mass-split fermion modes are solutions of the spectral problem $L^A \partial_A \Psi = \Psi \mathcal{M}$, where A ranges over the multi-indices of the 12 additional coordinates: 0 (the scalar), 12, 23, ..., 1234. Because the generalized mass operator $\mathcal{M}$ contains multivectors of different types, it possesses a mass hierarchy from the outset. We show how to construct the Gell-Mann matrices on the three-dimensional space of color modes (sectors). Unlike the quarks, the leptonic electroweak set employs a color-independent imaginary unit $J_\ell = (e_{12} + e_{23} + e_{31})/\sqrt{3}$, which excludes the lepton modes from strong interactions.

In our view, the conventional complexification of the Clifford algebra conceals the different nature of the quantum operators. We show that the electromagnetic, weak, color, generation, and Dirac complex structures may have different left or right geometric origins.

References

[1] W. Lu, *Clifford algebra Cl(0,6) approach to beyond the standard model and naturalness problems*, Int. J. Geom. Methods Mod. Phys., 21, 2450089, 2024.

[2] N. Gresnigt, *Electroweak structure and three fermion generations in Clifford algebra with S_3 family symmetry*, arXiv:2601.07857, 2026.

[3] C. Betancourt, *An electroweak theory of leptons in six dimensions*, Nucl. Phys. B, 1017, 116947, 2025.

[4] G. M. Quinta, *Spacetime Grand Unified Theory*, arXiv:2507.11564, 2025.

[5] M. Pavšič, *Spin gauge theory of gravity in Clifford space: A realization of Kaluza-Klein theory in 4-dimensional spacetime*, Int. J. Mod. Phys. A, 21, 5905, 2006.

[6] A. Ketsaris, *Generalized equation of relativistic quantum mechanics in a gauge field*, arXiv:quant-ph/9909087, 1999.

[7] D. Hestenes, *Space-time structure of weak and electromagnetic interactions*, Found. Phys., 12, 153, 1982.

Operational quantum reconstruction, Feynman rules, composition algebras, and beyond

Jens Köplinger
Greensboro, NC, USA
jenskoeplinger@gmail.com

As part of the Quantum Reconstruction Program, we recently explored [1] an operational model for transition amplitudes between measurements [2]. The associative real composition algebras, (split-)complex and (split-)quaternion, showed utility towards answering the question of probabilities from the envisioned building block of quantum reality. This talk reports on algebras that become permissible under a seemingly minute generalization, as sketched in the outlook, which includes Clifford algebras used in contemporary modeling of the algebraic structure of the Standard Model of particles.

References

[1] J. Köplinger, M. Habeck, P. Goyal, Int. J. Theor. Phys., 65, 128, 2026.

[2] P. Goyal, K. H. Knuth, J. Skilling, Phys. Rev. A, 81, 022109, 2010.

A spacetime algebra approach to the symmetry groups of particle physics

A.N. Lasenby
University of Cambridge, Cambridge, UK
a.n.lasenby@mrao.cam.ac.uk

The spacetime algebra (STA) provides a unified, geometric-algebra treatment of electromagnetism, gravity and quantum mechanics [1,2]. Since we observe only four spacetime dimensions of known signature, it is natural to ask whether the STA alone, with no additional mathematical structure, might also underlie the gauge symmetries of particle physics.

Recent work by Furey and others [3,4] has revealed a striking correspondence between the octonions — the largest of the normed division algebras — and the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$. In earlier work [5] the author showed that this octonionic structure can be realised entirely within the STA, with octonionic products expressed as ordinary geometric products of Dirac spinors.

This talk extends that correspondence through the use of *sandwich products* in the STA, which give a direct and computationally transparent route between octonionic 'chains' and standard Clifford/matrix algebra. We show how the Standard Model gauge groups arise from the preservation of octonionic norms under distinct classes of STA sandwich transformation, and indicate how extension to the sedenions (the 16-dimensional Cayley–Dickson algebra) opens a route towards grand-unified structures such as those of Georgi, and of Pati and Salam.

The overall picture is one in which the geometric algebra of observed spacetime, with no further postulated structure, already contains the algebraic scaffolding of the Standard Model.

References

[1] D. Hestenes, *Space-Time Algebra*, Gordon and Breach, New York, 1966.

[2] C. Doran, A. Lasenby, *Geometric Algebra for Physicists*, Cambridge University Press, Cambridge, 2003.

[3] C. Furey, *Standard model physics from an algebra?*, arXiv:1611.09182, 2018.

[4] C. Furey, *Three generations, two unbroken gauge symmetries, and one eight-dimensional algebra*, Phys. Lett. B, 785, 84-89, 2022.

[5] A. Lasenby, *Some recent results for SU(3) and octonions within the geometric algebra approach to the fundamental forces of nature*, Math. Methods Appl. Sci., 45, 4988-5032, 2022.

The biquaternionic Lanczos equation (1929) as a non-standard point of view on the Dirac equation

Nikolay Marchuk
Steklov Mathematical Institute, Department of mathematical physics, Russia
nmarchuk@mi-ras.ru

In 1929, Cornelius Lanczos proposed an equation for the electron using biquaternions (complexified quaternions). To date (2026), the Lanczos equation has not entered the mainstream of mathematical and theoretical physics. I believe that the Lanczos equation initiates new directions for modifications and generalizations of the Dirac equation for the electron. These areas are discussed in the report.

References

[1] C. Lanczos, *Die tensoranalytischen Beziehungen der Diracschen Gleichung (The tensor analytical relationships of Dirac's equation)*, Zeits. f. Phys. 57, 447-473, 1929.

[2] A. Gsponer, J. Hurni, *Lanczos's equation to replace Dirac's equation?*, In: J.D. Brown et al., eds., Proceedings of the Cornelius Lanczos International Centenary Conference Raleigh, North Carolina, December 12-17, 1993.

[3] N. G. Marchuk, *Conservative equations in field theory: conservation and symmetry laws*, Differ. Equ., 61:6, 872–892, 2025.

[4] N. G. Marchuk, *Conservative field equations and scalar fields (equations for leptons)*, arXiv: 2604.12822, 2026.

Massive electrons, massless neutrinos: Electroweak gauge fields and a Higgs mechanism from a single Dirac-like equation in $\mathcal{G}_{4,1}$

Gene E. McClellan, Ian J. McClellan
Applied Research Associates, Inc., USA
gmcclellan@ara.com

Geometric algebra lets electroweak theory, the particle states, the gauge fields, and the origin of mass, be written in one associative algebra with one first-order field equation. States and operators live in the same algebra rather than in separate, index-laden spinor and matrix spaces. Working in $\mathcal{G}_{4,1}$ (four spatial and one temporal dimension), we apply the gauging procedure of the Standard Model (SM) directly to a Dirac-like equation and obtain both Abelian $U(1)$ and non-Abelian $SU(2)$ gauge fields as well as a mass term for the electron. Local symmetry produces a distinctive "sandwich" coupling, with the vector gauge field acting from the left of the field and the Lie-algebra generator from the right. The framework yields a compact Higgs mechanism. A single change to minimal coupling, complementing the vector gauge field with a constant scalar Higgs ground-state value and embedding a left-ideal projector in the mass term, places the electron and neutrino in complementary left ideals of $\mathcal{G}_{4,1}$. The electron then acquires its mass from the Higgs coupling while the neutrino remains massless, both fields arising as solutions of one and the same field equation.

This work builds on a Dirac-like equation in $\mathcal{G}_{4,1}$ [1,2] whose plane-wave solutions reproduce the chiral electron and neutrino states of an electroweak Fock-space ladder (nilpotent operators with canonical anticommutation relations), a geometric-algebra realization of Furey's $\mathbb{C}(4)$ construction [3] carrying full Lorentz-invariant space and time dependence. That earlier equation assigned both particles the same mass; the construction here removes that limitation. The Higgs representation is deliberately a single-component (unitary-gauge) model, and a full $SU(2)_L \times U(1)_Y$ doublet treatment with symmerty breaking remains future work, but the result shows how one geometric-algebra equation can carry states, gauge interactions, and a mass mechanism together while making the chiral and ideal structure of the electroweak sector manifest.

References

[1] G.E. McClellan, *Operators and field equations in the electroweak sector of particle physics*, Adv. Appl. Clifford Algebras, 31, 65, 2021.

[2] G.E. McClellan, *Using raising and lowering operators from geometric algebra for electroweak theory in particle physics*, Adv. Appl. Clifford Algebras, 29, 90, 2019.

[3] C. Furey, *A demonstration that electroweak theory can violate parity automatically (leptonic case)*, Int. J. Mod. Phys. A, 33, 1830005, 2018.

Spinor Representations from Real Geometric Invariants of Spin Groups

Martin Roelfs[1], **Imke Verlinden**[2], **David Eelbode**[3]
University of Antwerp, Belgium
martin.roelfs@uantwerpen.be

When it comes to the construction of spinors spaces in Clifford algebra's $\mathrm{Cl}_{p,q}(\mathbb{R})$, there have long been three different cultures. The first, tracing its roots back to Élie Cartan, proceeds by first complexifying to $\mathrm{Cl}_{p,q}(\mathbb{C})$, finding the Cartan subalgebra of commuting spin generators, and using the set of eigenvectors to the Cartan subalgebra to construct an idempotent which defines the spinor space. The second culture, developed by Crumeyrolle, Ablamowicz, and Lounesto among others, does not insist the idempotent must come from eigenvectors of the Cartan subalgebra, but instead notices that $\mathrm{Cl}_{p,q}(\mathbb{R})$ already has sufficient idempotents to form the spinor space once you know the matrix-algebra type of the spinor space. This approach therefore no longer requires complexification, although it loses the clear relationship to the spin group that the first approach had. The third culture, due to David Hestenes, does not require an idempotent at all, and instead treats spinors as elements of the even subalgebra $\mathrm{Cl}^+(\mathbb{R})$ that act grade-preservingly on vectors.

Though formally isomorphic, these constructions are not interchangeable.
In order to bridge the gap between the cultures, we introduce a geometric methodology for spinor representations based on invariant structures of Spin groups. The methodology starts the same way as the widely accepted construction due to Cartan, but by focussing on the invariant structures it manages to obtain the spinor space directly within $\mathrm{Cl}_{p,q}(\mathbb{R})$, thereby avoiding complexification entirely. It highlights how spinors are characterized by geometric invariants of $\mathrm{Spin}(p, q)$ transformations, and illuminates why spinors satisfy what some authors call the multivector condition [3]: the fact that spinors transform one-sidedly under $\mathrm{SU}(n)$. By following in the footsteps of Cartan, our methodology keeps the geometric foundation of the spinor representation explicit while simultaneously illustrating why complexification both works so well and yet is entirely redundant.
The real interpretation opens up the possibility to describe the geometry of fermionic degrees of freedom, the geometric interpretation of Weyl and Majorana conditions, and connections with the Cayley–Dickson construction.

References

[1] J.C. Baez, *The octonions*, Bulletin of the American Mathematical Society, 39(2), 145–205, 2002.

[2] J. Polchinski, *Volume 2: Superstring Theory and Beyond*. Cambridge University Press, 1998.

[3] N. Furey, *An algebraic roadmap of particle theories, Part I: General construction*, Annalen der Physik, 537(4), 2400322, 2025.

[4] C. Doran, A. Lasenby, *Geometric algebra for physicists*. Cambridge University Press, 2003.

[5] M. Roelfs, S. De Keninck, *Graded Symmetry Groups: Plane and Simple* Adv. Appl. Clifford Algebras 33, 30, 2023.

[6] D. Eelbode, M. Roelfs, S. De Keninck, *Outer and Eigen: Tangent Concepts*, Adv. Appl. Clifford Algebras 35, 52, 2025.

[7] M. Roelfs, D. Eelbode, *Lepton Triptych I: Geometric Foundations of Electroweak Symmetry in the Real Clifford Algebra $Cl_4(\mathbb{R})$*, arXiv:2510.13834, 2025.

On the Dirac-Hestenes equation in higher dimensions and its properties

Rumyantseva Sofia, Shirokov Dmitry
HSE University, Russia
srumyanceva@hse.ru

This talk is devoted to a higher-dimensional generalization of the Dirac–Hestenes equation in the geometric algebra $\mathcal{C}\ell(1, n)$. This equation can be viewed as a real formulation of the Dirac equation. The Dirac–Hestenes formalism provides a deeper geometric interpretation of physical phenomena, since the associated wave function is entirely real-valued. In the classical case, $n = 3$, the Dirac–Hestenes equation is known to be equivalent to the conventional Dirac equation. Solutions of one equation can be obtained directly from solutions of the other. The talk demonstrates that this correspondence extends naturally to the multidimensional case. Special attention is paid to the dependence on the parity of the dimension n. The even- and odd-dimensional cases exhibit different matrix representations and therefore require separate investigation. Moreover, in even dimensions, there are two types of spinors which are solutions to the Dirac equation: semi-spinors and double spinors. Therefore, we consider three different types of spinors.
The symmetry properties of the multidimensional Dirac–Hestenes equation are also discussed. It is shown that the equation exhibits both gauge invariance and Lorentz invariance.

References

[1] S. Rumyantseva, D. Shirokov, *Introducing Multidimensional Dirac-Hestenes Equation*, Advances in Applied Clifford Algebras, 35(3), Article 24, 2025.

[2] S. Rumyantseva, D. Shirokov, *Lorentz Invariance of the Multidimensional Dirac-Hestenes Equation*, Advances in Applied Clifford Algebras, 36(1), Article 5, 2026.

Octonions, and an $E_8 \times E_8$ theory of unification

Tejinder P. Singh
Tata Institute of Fundamental Research, India
tpsingh@tifr.res.in

Over the last decade or so, we have been seeking a reformulation of quantum (field) theory which does not depend on external classical time. Such a reformulation is essential at all energy scales, not just at the Planck scale. The reason being that classical (space-)time exists only when the universe is dominated by macroscopic classical bodies. However, in principle, even at low energies, a physical universe need not have anything classical about it. Hence, when everything is 'quantum', the aforesaid reformulation is required. From the vantage point of this reformulation, the current formulation of quantum theory becomes a (highly successful) approximation, valid only when the universe is dominated by macroscopic classical bodies, as our actual universe is. The reformulation is a more exact description of quantum phenomena and of quantum spacetime geometry, and remarkably, it also turns out to be a candidate theory for the unification of fundamental forces [1, 2].

There are five key mathematical ingredients which go into the development of such a reformulation. These are : normed division algebras, Adler's theory of Trace Dynamics, Connes' non-commutative geometry and the accompanying spectral action principle, Clifford algebras, the exceptional Lie groups and the exceptional Jordan algebra. Every one of these five ingredients plays a distinct role in the reformulation of quantum theory and the accompanying unification.

Every spacetime point (which of course is real-number valued) is replaced by a non-commutative structure, and in this we are guided by division algebras. There are only four such algebras, and besides the real numbers and complex numbers, they are the quaternions (non-commutative) and the octonions (non-commutative and non-associative). Our research shows that it pays to replace every point by a copy of the 16-dim split bioctonion. This replacement enables a structure (scaffolding) rich enough to incorporate internal symmetry space alongside a pre-spacetime, as well as three generations of chiral fermions of the standard model [3, 4].

On this pre-spacetime, at every bioctonionic 'point', we overlay matter and field degrees of freedom, described by matrices/operators. These obey the rules of Adler's trace dynamics: a matrix-valued Lagrangian dynamics, which turns out to be more general than quantum theory, and is hence a pre-quantum dynamics. Each matrix has sixteen matrix-valued components, one component per octonionic direction. This generalises, in a rather natural way, the four real-number valued components of the position vector of a classical particle living on a 4-dim spacetime. Every point of the 4-dim spacetime has been replaced by the 16-dim split bioctonion, and the 4-dim position vector of the particle has been replaced by the 16-dim matrix-valued dynamical variable. We call each such entity (bioctonion + dynamical variable) an atom of spacetime-matter (STM atom), and it has the action given as

$$\frac{S}{\hbar} = \frac{1}{2} \int \frac{d\tau}{\tau_{Pl}} \, Tr \left[\frac{L_P^2}{L^2} \dot{Q}_1^\dagger \dot{Q}_2 \right]$$

This is the pre-quantum, pre-spacetime generalisation of the action of a classical free particle. Here, τ is a novel, new time parameter, the Connes time, whose existence is a consequence of the Tomita-Takesaki theorem for von Neumann algebras, relevant to Connes' non-commutative geometry. And in our present case, we have an instance of a non-associative, non-commutative geometry. Connes' time is over and above, and extrinsic to, the 16-dim split bioctonionic space: the latter evolves in Connes' time, and the presence of STM atoms 'curves' this space, in the same sprit in which matter in general relativity curves Minkowski spacetime.
How do we guarantee that the dynamical variables Q_1, Q_2 are related to the physical world that we presently observe? This is ensured by an application of the Chamseddine-Connes spectral action principle, according to which, the eigenvalues of the Dirac operator serve as dynamical variables for general relativity. The Einstein-Hilbert action for gravity, with Yang-Mills fields included, can be expressed as a sum over squares of eigenvalues of the Dirac operator. Then, in the spirit of trace dynamics, each eigenvalue is raised to the status of an operator. This operator is the Dirac operator on the bioctonionic space - being essentially the octonionic gradient operator [2]. Fermionic degrees of freedom are also included in this (generalised) Dirac operator, and the variables Q_1, Q_2 are identified with the Dirac operator. Thus, one original eigenvalue corresponds to one copy of the Dirac operator (an STM atom) in the trace dynamics. This is how a pre-quantum, pre-spacetime theory is arrived at. When a large number of STM atoms entangle because of interactions, a quantum-to-classical transition takes place: operator-valued dynamical variables are mapped back to one or the other of their eigenvalues, and thereby classical spacetime and classical matter and gauge fields are recovered. Those STM atoms which do not get critically entangled are described by trace dynamics, or (to a very good) approximation, by quantum field theory on the emergent classical spacetime background.
The universe is assumed to have begun as a collection of enormously many STM atoms, each of which (by way of the action written above) obeys an $E_8 \times \omega E_8$ symmetry. The universe is assumed to have undergone an inflation-like de Sitter expansion up until the electroweak scale, when each of the E_8 branches as follows:

$$\begin{aligned} E_{8L} \; & \rightarrow SU(3)_L^{geom} \times E_{6L} \xrightarrow{E_{6L}} SU(3)_c \times SU(3)_{F,L} \times SU(3)_L \\ & \xrightarrow{SU(3)_L} SU(2)_L \times U(1)_{\gamma_1} \rightarrow U(1)_Y \\ E_{8R} \; & \rightarrow SU(3)_R^{geom} \times E_{6R} \xrightarrow{E_{6R}} SU(3)_{c'} \times SU(3)_{F,R} \times SU(3)_R \\ & \xrightarrow{SU(3)_R} SU(2)_R \times U(1)_{\gamma_2} \rightarrow U(1)_{Ydem} \end{aligned}$$

Of the three $SU(3)$s arising from the branching of E_{6L}, the $SU(3)_c$ implements the color gauge symmetry of QCD. Furthermore, $SU(3)_{F,L}$ is the non-gauged global flavor symmetry which is responsible for three left-handed fermion generations [5] described by the

exceptional Jordan algebra $J_3(\mathbb{O}_c)$. $SU(3)_L$ branches as $SU(2)_L \times U(1)_Y$ giving rise to the electroweak sector, and the $SU(2)_L$ acts only on left-handed particles.
As regards the branching of the second E_6, which we label E_{6R}, the $SU(3)_{c'}$ symmetry is preferably not gauged (so as to be consistent with phenomenology) - it is global and explicitly broken at the electroweak scale. The $\omega SU(3)_{F,R}$ is the global flavor symmetry which describes three generations of right-handed fermions. The $SU(3)_R$ branches as $SU(2)_R \times U(1)_{Ydem}$. Here, the spontaneously broken $SU(2)_R$ gives rise to general relativity, and the unbroken $U(1)_{dem}$ arising from $U(1)_{Ydem} \to U(1)_{dem}$ is a new force, dubbed dark electromagnetism. It is sourced by the square-root of mass and can be made cosmologically and phenomenologically safe. The emergence of general relativity in this manner, and the preceding gravi-weak unification, was briefly discussed in [6] and is analysed in detail in a companion work [7]. The $SU(3)_L^{geom}$ and $\omega SU(3)_R^{geom}$ together give rise to the 16-dim split bioctonionic space, from which spacetime and internal symmetry space are emergent [4].
The triality of $Spin(8) \in E_6$ explains why there are three fermion generations. Complex split bioctonions generate the Clifford algebra $Cl(7)$, which is isomorphic to $Cl(6) \oplus Cl(6)$. The two copies of $Cl(6)$ are used to construct spinors as minimal left ideals describing one generation of SM chiral fermions. The breaking of triality and of left-right symmetry gives rise to the observed fermion mass-hierarchy [5].

References

[1] T. P. Singh, *Trace dynamics, octonions and unification: An $E_8 \times E_8$ theory of unification*, J. Phys. Conf. Ser., 2912, 012009, 2024.

[2] S. Farnsworth, F. Finster, C. F. Paganini, T. P. Singh, *Causal Fermion Systems, Non-Commutative Geometry and Generalized Trace Dynamics*, arXiv:2603.05018, 2026.

[3] V. Vaibhav, T. P. Singh, *Left-Right Symmetric Fermions and Sterile Neutrinos from Complex Split Biquaternions and Bioctonions*, Adv. Appl. Clifford Algebras, 33, 2023.

[4] Singh, T. P. *Spacetime and Internal Symmetry from Split Bioctonions and the Two Extra SU(3)'s of $E_8 \times \omega E_8$*, Preprints, 2025100437, 2024.

[5] T. P. Singh, *Fermion mass ratios from the exceptional Jordan algebra*, arXiv:2508.10131.

[6] P. Kaushik, V. Vaibhav, T. P. Singh, *An $E_8 \otimes E_8$ unification of the standard model with pre-gravitation, on an exceptional Lie-algebra valued space*, arXiv:2206.06911.

[7] P. S. Wesley, T. P. Singh, J. M. Isidro, *Gravity and electroweak sector from symmetry breaking of an $SO(3,3)$ BF theory*, arXiv:2602.19151.

Set of Clifford algebras and a Hydrogen atom solution with an improved Dirac equation

Thierry Socroun
GARFE, France
tsocroun@yahoo.fr

This paper aims to construct a set of Clifford algebras (noted C_n) that are simple and usefull. The main idea is that every C_n is a physical one (unlike other constructions) and it minimizes noncommutativity. To get it, we add at each step, not one but two new independent Clifford symbols that correspond to a new generation of Pauli matrices.
To show how this construction is useful, we use it to solve the Hydrogen atom within the framework of an improved Dirac equation.

References

[1] C. Daviau, J. Bertrand, *Resolution in the case of the hydrogen atom of an improved Dirac equation*, J. of Mod. Phys., 11:1075-1090, 2020.

[2] H. Krüger, *New solutions of the Dirac equation for central fields.* In: D. Hestenes and A. Weingartshofer, editors, The Electron. Kluwer, Dordrecht, 1991.

[3] D. Hestenes, *Real Spinor Fields*, Journal of Mathematical Physics, 8(4), 798-808, 1967.

Spinor approach to guiding center approximation

Václav Zatloukal, Vojtěch Vanc
Czech Technical University in Prague, Faculty of Nuclear Sciences and Physical Engineering, Czech Republic
vaclav.zatloukal@fjfi.cvut.cz

Spinors were introduced to physics to describe the behaviour of quantum particles in inhomogeneous magnetic fields (the Stern-Gerlach experiment). Since then they have become an integral part of the formalism of quantum mechanics and quantum field theory, commonly realized as $\mathbb{C}^2$ vectors in non-relativistic theories, and $\mathbb{C}^4$ vectors in relativistic theories. Working with Clifford's geometric algebra (of space or spacetime), it is possible, and indeed convenient, to represent spinors as even multivectors [1], [2]. This representation is free of complex numbers, and so it provides spinors with clearer geometric interpretation — in particular, in the non-relativistic case, they are identified as scaled rotors, i.e., configurations of an orthonormal frame (that include a magnitude parameter).

Geometric algebra understands spinors first and foremost as geometric objects (rather than mysterious quantum entities); objects that are useful not only in quantum, but also in classical domain. The geometric algebra spinors were developed by Hestenes, and employed in his classical interpretations of the Dirac equation of relativistic quantum mechanics, whereby the quantum electron is viewed as a charge that oscillates around a center of mass [3]. The insights of geometric algebra were therefore used to understand certain aspects of quantum theory in classical terms.

In our work we employ geometric algebra spinors in purely classical physics setting, namely, to analyse the motion of a charged particle in a slowly varying electromagnetic field. In a homogeneous magnetic field, the particle's trajectory forms a helix whose axis is parallel to the magnetic field. When weak inhomogeneities are present, the helix bends. This situation is encountered and analysed in plasma physics, where the so-called *guiding center approximation* is commonly employed [4]. It decomposes the particle motion into a fast gyro-motion around a fictitious guiding center position, and a slow motion of the guiding center along the magnetic field lines. The guiding center does not follow the field lines perfectly, but its velocity is corrected by additional *drifts* (the curvature drift, the grad-B drift corresponding to the Stern-Gerlach force, and other corrections). The traditional approach is based on a two-scale analysis of the Lorentz equation for the particle velocity. On the fast timescale the particle performs fast oscillations (the gyro-motion), while on the slow timescale (when we "zoom out") we observe only the motion of the guiding center.

Both, the fast and the slow motion, can captured by a single spinor variable. Therefore, in our approach to the guiding center approximation we start from the spinorial version of the Lorentz equation [1]. This equation is in many aspects similar to the quantum Schrödinger equation for the time evolution of a spin degree of freedom, which allows us to exploit the quantum formalism in our purely classical setup. We carry out a standard perturbation analysis of quantum theory, that is, set up the interaction picture (with derivatives of the magnetic field being the small perturbations), write the Dyson expansion, and, finally, extract the guiding center drifts from the ensuing approximative solution of the spinor equation.

Thus, we demonstrate another concrete application of spinors in classical physics, taking advantage of quantum techniques, but avoiding speculations about interpretations of the quantum theory itself. At the same time, we do believe that our analysis provides certain insights into the way, in which spinors (or entities that are naturally described by spinors) interact with the electromagnetic field.

References

[1] C. Doran, A. Lasenby, *Geometric algebra for physicists*. Cambridge Univ. Press, 2004.

[2] V. Zatloukal, *Real spinors and real Dirac equation*, Adv. Appl. Clifford Algebras, 32, 45, 2022.

[3] D. Hestenes, *Gyromagnetics of the Electron Clock*, IEEE Access, vol. 13, pp. 53772-53803, 2025.

[4] H.J. de Blank, *Guiding Center Motion*, Fusion Science and Technology, 45:2T, 47-54, 2004.

Differential invariants and geometric complexes

This session explores recent progress at the intersection of Clifford analysis, representation theory, and differential geometry. Modern Clifford analysis offers a local function theory for invariant first-order systems of PDEs and reveals rich structures such as Howe duality and geometric complexes of invariant differential operators. Symmetries of invariant operators appear in a broad scale of incarnations.

Session organisers: Vladimir Soucek (Charles University, Czechia), Roman Lavicka (Charles University, Czechia), Jan Slovak (Masaryk University, Czechia).

Weitzenböck remainder spectrum on rational homogeneous varieties

Eder M. Correa, Lucas Almeida, Samuel Wainer
`University of Campinas - UNICAMP, Brazil`
`lalmeida@ime.unicamp.br`

In this work, we precisely describe the spectrum of closed invariant $(1, 1)$-forms viewed as an operator acting on complex spinor bundles over rational homogeneous varieties. Using this result, we describe the spectrum of the Weitzenböck remainder of Spin^c Dirac operators on rational homogeneous varieties. In particular, we present an explicit formula for their smallest eigenvalue. As a byproduct, we obtain a new lower bound for the eigenvalues of the Spin^c Dirac operator, expressed in terms of Lie-theoretic data. Additionally, combining the Atiyah-Singer index theorem with the Borel-Weil-Bott theorem, we provide a complete classification of Spin^c structures on rational homogeneous varieties which admit harmonic spinors. In this last setting, we present an explicit formula for the index of the associated Spin^c Dirac operator in terms of Lie theory.

References

[1] H. Azad, I. Biwas, *Quasi-potentials and Kähler-Einstein metrics on flag manifolds, II*, J. Algebra, 269(2):480-491, 2003.

[2] D.V. Alekseevsky, I. Chrysikos, *Spin structures on compact homogeneous pseudo-Riemannian manifolds*, Transformation Groups, 4(3):659-689, 2019.

[3] M. Atiyah, I. Singer, *The index of elliptic operators, III*, Annals of Mathematics, 87(3):546-604, 1968.

[4] E. Correa, *Deformed hermitian Yang-Mills equation on rational homogeneous varieties*, arXiv:2304.02105, 2023.

[5] O. Hijazi, *Eigenvalues of the Dirac Operator on Compact Kähler Manifolds*, Communications in Mathematical Physics, 1994.

[6] N. Hitchin, *Harmonic spinors*, Advances in Mathematics, 14(1):1-55, 1974.

[7] H. Lawson, M. L. Michelsohn, *Spin Geometry (PMS-38)*, Princeton University Press, Volume 38, volume 20, 2016.

[8] R. Nakad, M. Pilca, *Eigenvalue estimates of the spin^c Dirac Operator and Harmonic Forms on Kähler-Einstein Manifolds*, SIGMA. Symmetry, Integrability and Geometry: Methods and Applications, 11:054, 2015.

Modified Dirac operators and transformations betweenharmonic and twistor spinors

Ümit Ertem
Ankara University, Turkey
umitertem@ankara.edu.tr

Symmetry operators of harmonic spinors and twistor spinors can be constructed from conformal Killing-Yano forms which are antisymmetric generalizations of conformal Killing vector fields[1]. Special types of potential forms generalizing the solutions of the Laplace equation are used in the construction of transformation operators relating harmonic spinors to twistor spinors [2]. We consider the modifications of connections and Dirac operators in terms of various degree differential forms and investigate the effects of these modifications to the constructions of symmetry operators and transformation operators of modified harmonic and twistor spinors. Special cases of these modifications include the gauge potential and supergravity flux generalizations of connections and Dirac operators [3]. We find the conditions to generalize the construction of modified symmetry and transformation operators in terms of modified conformal Killing-Yano forms on more general manifolds admitting gauge and flux generalizations of geometric operators.

References

[1] Ü. Ertem, Ö. Kelekçi, Ö. Açık, *Algebra structure of conformal Killing-Yano forms in geometries with skew-symmetric torsion*, Classical and Quantum Gravity, 42, 215001, 2025.

[2] Ü. Ertem, *Harmonic spinors from twistors and potential forms*, Journal of Mathematical Physics,59, 112301, 2018.

[3] Ü. Ertem, *Symmetries of modified Dirac operators in supergravity flux backgrounds*, NuclearPhysics B, 1007, 116665, 2024.

Complexes for the Dirac operator in several even-dimensional variables

Lukáš Krump
Charles University, Mathematical Institute, Czech Republic
krump@karlin.mff.cuni.cz

In the theory of several Clifford variables, we consider the operator $D_k = (\partial_{\underline{x}_1}, \ldots, \partial_{\underline{x}_k})$, whose null-solutions are monogenic (spinor-valued) functions of k variables in $\mathbb{R}^{2n}$. To understand this operator, we do not look for its null-solutions (i.e. its kernel), but for its image. This image can be realized as a kernel of a suitable differential operator D_k^1. Continuing in the same way, we obtain a sequence D_k^j, $j = 1, \ldots, t$, of operators which form a resolution (or a generalized Dolbeault sequence) of the first operator $D_k = D_k^0$. There are essentially two possible cases: if $k \leq n$, these are so-called stable cases, which are quite understood and-well described. The non-stable cases ($k > n$) are much less understood. A survey of known results will be given, and a construction (using the Penrose transform method) of the resolutions in a few low-dimensional non-stable cases will be described in the contribution.

References

[1] L. Krump, *A Resolution for the Dirac Operator in Four Variables in Dimension 6*, Adv. Appl. Clifford Alg., 19, pp. 365–374, 2009.

[2] L. Krump, *The generalised Dolbeault complex for four Dirac operators in the stable rank*, Proceedings of the ICNAAM, Kos 2008, AIP Conference Proceedings, pp. 670-673, 2008.

[3] L. Krump, T. Salač, *Exactness of the generalized Dolbeault complex for k Dirac operators in the stable rank*, Numerical Analysis and Applied Mathematics ICNAAM 2012, Melville, pp. 300-303, 2012.

Quantum Clifford analysis

Andrey Krutov[1], **Roman Lávička**[2], **Vladimír Souček**[3]
Charles University, Mathematical Institute, Czech Republic
andrey.krutov@matfyz.cuni.cz

We develop a version of the quantum deformation of the basic blocks of Clifford analysis – the Howe dual pair and the Fischer decomposition for spinor valued functions. We construct the q-deformed Clifford algebra $\mathrm{Cl}_q(V)$ of the standard representation V of the quantum group $U_q(\mathfrak{o}_N)$. Then the quantum Dirac operator is defined as an element of the braided tensor product of $\mathrm{Cl}_q(V)$ and the quantum analogue of the algebra of differential operators on V with polynomial coefficients constructed in [1]. The decomposition of the quantum analogue of polynomial functions on V based on harmonic polynomials was obtained in [1]. Using the quantum Dirac operator we extended this decomposition to the case of the quantum analogue of spinor-valued polynomials based on monogenic spinor valued polynomials.

References

[1] T. Sugitani, *Harmonic analysis on quantum spheres associated with representations of $U_q(\mathfrak{so}_N)$ and q-Jacobi polynomials*, Compositio Math., 99, no. 3, 249-281, 1995.

Twistor complexes in symplectic spin geometry

Svatopluk Krýsl
Charles University, Mathematical Institute, Czech Republic
krysl@karlin.mff.cuni.cz

Symplectic spin geometry is a study of symplectic connections on symplectic manifolds which admit the so called metaplectic structure (Kostant [1]), that corresponds to the spin-structure on Riemannian and pseudo-Riemannian manifolds.
The connections induce symplectic Dirac operators, introduced by Habermann [2], symplectic twistor operators, and their higher spin analogues. We introduce these operators in the context of representation theory of the symplectic group and focus on the twistor operators and cases when these operators form differential complexes, on their ellipticity and Lefschetz property.

References

[1] B. Kostant, *Symplectic spinors*, Symposia Mathematica, Vol. XIV (Convegno di Geometria Simplettica e Fisica Matematica, INDAM, Rome, 1973), pp. 139 - 152, Academic Press, London, 1974.

[2] K. Habermann, *The Dirac operator on symplectic spinors*, Ann. Global Anal. Geom., 13, no. 2, Springer-Verlag, 155-168.

Separation of variables in non-stable range

Roman Lávička
Charles University, Mathematical Institute, Czech Republic
lavicka@karlin.mff.cuni.cz

In this talk, we focus on the separation of variables for scalar-valued and spinor-valued polynomials in k variables on the Euclidean space of dimension m, with its natural symmetry under the action of the orthogonal group $O(m)$. We first recall the decomposition of polynomials into products of invariant polynomials and solutions in the stable range (i.e., when $m \geq 2k$); see, e.g., [2,3]. We then present recent breakthrough results for the scalar-valued case in the non-stable range; see [1].

References

[1] D. Beďatš, *Separation of variables for scalar-valued polynomials in the non-stable range*, Journal of Algebra, 51, 281-304, 2024.

[2] R. Lávička, V. Souček, *Fischer decomposition for spinor valued polynomials in several variables*, arXiv:1708.01426, 2017.

[3] R. Lávička, *Separation of variables in the semistable range.* In: Topics in Clifford Analysis: Special Volume in Honor of Wolfgang Sprößig, Springer International Publishing, Cham, Germany, pp. 395–403, 2019.

The S-spectrum and S-resolvent estimates for spin Dirac operators on manifolds with boundary

Simão Lucas
Polytechnic University of Milan, Italy
simao.lucas@polimi.it

In this talk, we discuss the spectral theory based on the S-spectrum of spin Dirac operators on manifolds with boundary.
Since the S-spectrum and the S-resolvent are associated with a second-order polynomial in the Dirac operator D, that is, with the operator defined as $Q_s(D) := D^2 - 2s_0 D + |s|^2$, we can associate to the Dirac operator boundary conditions that can be of Dirichlet type but also of Robin-like type and study elliptic boundary value problems in the weak sense. We start by discussing some concrete examples. Later, we present results for the general case of an arbitrary compact spin manifold with boundary.
This is joint work with I. Beschastnyi (Centre Inria d'Université Côte d'Azur, France), F. Colombo (Politecnico di Milano, Italy), and I. Sabadini (Politecnico di Milano, Italy).

On grade automorphism and Lie groups in ternary and generalized Clifford algebras

Dmitry Shirokov
HSE University, Moscow, Russia; Institute for Information Transmission Problems of the Russian Academy of Sciences, Moscow, Russia
dshirokov@hse.ru

We introduce the concept of a grade automorphism in generalized Clifford algebras (in particular, ternary Clifford algebras) and discuss a number of its properties. This operation, while not an involution, naturally generalizes the classic grade involution (main involution) in ordinary (quadratic) Clifford geometric algebras. The corresponding Lie groups and Lie algebras in ternary and generalized Clifford algebras are studied. The proposed operation is expected to find applications across various fields that utilize ternary and generalized Clifford algebras, including analysis, physics, engineering, and computer science. The results further develop the investigations initiated in [1, 2, 3].

References

[1] D. Shirokov, *On Grade Automorphism in Ternary Clifford Algebras*, Hypercomplex Analysis and Its Applications. ICHAA 2024, Trends in Mathematics, 9, eds. Faustino, N., Ferreira, M., Kähler, U., Vieira, N., Birkhäuser, Cham, 143–150, 2025.

[2] D. Shirokov, *On Unitary Groups in Ternary and Generalized Clifford Algebras*, Advances in Applied Clifford Algebras, 35, 25, 2025.

[3] D. Shirokov, *On Inner Automorphisms Preserving Fixed Subspaces of Clifford Algebras*, Advances in Applied Clifford Algebras, 31, 30, 2021.

On the kernel of the k-Cauchy-Fueter operator

Vladimír Souček
Charles University, Mathematical Institute, Czech Republic
ivanpombo@ua.pt

The study of the kernel of the Dirac operator D in several variables has a long tradition in Clifford analysis, see [1]. Something special is happening in dimension 4. It is due to the fact that the symmetry group of the operator D is suddenly much bigger. The whole resolutions starting with the k-Cauchy-Fueter operator were constructed by R. Baston in [2] and described explicitly in [2] and [4]. The kernel of the first operator was described explicitly in [5] using the Penrose transform technique.
The goal of the lecture is to use the big symmetry group of the operator D for an easy explicit construction of an orthogonal basis for the kernel of the operator D. The lecture is based on the common work with R Lávička and W. Wang.

References

[1] R. Baston, *Quternionic complexes*, J. Geom. Physics, 8, 29-52, 1992.

[2] J. Bureš, A. Damiano, I. Sabadini, *Explicit resolutions for several Fueter operators*, J. Geom. Phys., 57, 765-775, 2007.

[3] F. Colombo, I. Sabadini, F. Sommen, D. Struppa, *Analysis of Dirac systems and computational algebra*, Progress in Math. Physics,, vol. 39, Birkhauser, 2004.

[4] F. Colombo, V. Souček, D. Struppa, *Invariant resolutions for several Fueter operators*, J. Geom. Phys., 56, 1175-1191, 2006.

[5] Q. Kang, W. Wang, *On Penrose integral formula and series expansion of k-regular functions on the quaternionic space* $\mathbb{H}^n$, J. Geom. Physics, 64, 192-208, 2013.

Fractional integro-differential operators in two and higher dimensions

This session explores fractional integro-differential operator theory within Clifford algebras, focusing on nonlocal, multiscale methods for multivector-valued functions and fields. Core themes include fractional Dirac-type operators and their powers, fractional Clifford–Fourier transforms, semigroup and spectral constructions, and uncertainty principles for fractional Clifford transforms. Topics include, but are not limited to: Foundations of fractional Riemann–Liouville/Caputo/Riesz operators in Clifford settings; Fractional powers of Dirac, d-bar, and radial operators; Two-sided fractional Clifford–Fourier frameworks and kernel representations; Fractional pseudodifferential and Hankel–Clifford operator calculi; Fischer/CK extensions and functional calculi for nonlocal operators; Inversion, Plancherel, and mapping properties on Sobolev/Besov-type Clifford spaces; Discrete and semidiscrete fractional Dirac operators; Applications to nonlocal PDEs, imaging and inverse problems, time–frequency analysis, space-time modelling in mathematical physics, and artificial intelligence.

Session organisers: Milton Ferreira (Polytechnical University of Leiria, Portugal), Nelson Vieira (Universidade de Aveiro, Portugal).

Anisotropic fractional harmonicity

Antonio Di Teodoro
Deusto University, Spain
a.diteodorol@deusto.es

We introduce a framework of anisotropic fractional harmonicity based on semi-fractional Cauchy–Riemann operators combining classical and fractional derivatives in a directional way. This approach generates nonlocal and anisotropic analytic structures extending classical complex analysis.

We construct fractional Cauchy-type kernels and establish Green identities together with Cauchy and Cauchy–Pompeiu formulas in the Caputo setting. The theory naturally produces anisotropic fractional Laplacians, These operators generate a new class of anisotropic harmonic functions, revealing a rich interplay between locality, directionality, and fractional order. We also present recent progress toward the construction of an associated Poisson integral for this anisotropic fractional framework.

The classical theory is recovered as the fractional order approaches one. This framework provides new tools for the study of nonlocal operators and anisotropic fractional PDEs.

References

[1] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations*. Vol. 204, Elsevier, 2006.

[2] A. Fernandez, J.E., Restrepo, J.-D. Djida, *On the fractional Laplacian of a function with respect to another function*, Mathematical Methods in the Applied Sciences, 47(18), pp. 14079-14110. 2024.

[3] I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Vol. 198, Elsevier, 1998.

Space-fractional Dirac equations: Where Strichartz estimates meet gauge transforms

Nelson Faustino
University of Aveiro, Portugal
nfaust@ua.pt

In this talk, we bridge techniques from two major mathematical fields: the theory of singular integral operators, as developed by McIntosh [4], and the dispersive PDE theory for nonlinear Dirac-type equations, as pioneered by Escobedo and Vega [2], to present a unifying framework to study space-fractional Dirac equations in the linear and nonlinear cases. Starting with the pseudodifferential identity

$$D(-\Delta)^{\frac{\alpha-1}{2}} = (-\Delta)^{\frac{\alpha}{2}}\mathcal{H},$$

linking the Dirac operator D and the Riesz–Hilbert transform $\mathcal{H}$, we recast space-fractional Dirac equations as a coupled system of space-fractional Schrödinger equations. By leveraging this decomposition, we derive Strichartz-type estimates in homogeneous Sobolev spaces $\dot{H}^s$ and adapt Tao's gauge approach on the Benjamin-Ono equation [5] to establish a canonical correspondence between the linear and the associated nonlinear Dirac-type equations. Following the recent advances in Calderón-type inverse problems [3] and Paley-Wiener theory [1], our systematic approach offers a reliable path to study well-posedness of evolution PDEs in the context of Clifford analysis. If time permits, we will explore how these theoretical results extend to promising applications, such as shear-flow models in stratified hydrodynamics. We look forward to discussing how these connections may inspire future research in this direction.

References

[1] S. Bernstein, N. Faustino, *Paley-Wiener Type Theorems Associated to Dirac Operators of Riesz-Feller type*, J. Fourier Anal. Appl., 31, no. 4, 52, 2025.

[2] M. Escobedo, L. Vega, *A Semilinear Dirac Equation in $H^s(\mathbb{R}^3)$ for $s > 1$*, SIAM Journal on Mathematical Analysis, 28(2), 338-362, 1997.

[3] H. Quan, G. Uhlmann, *The Calderón problem for the fractional Dirac operator*, Math. Res. Lett., 31, no. 1, 279-302, 2024.

[4] A. McIntosh, *Clifford algebras, Fourier theory, singular integrals, and harmonic functions on Lipschitz domains.* In: Clifford algebras in analysis and related topics, pp. 33-87, CRC press, 1996.

[5] T. Tao, *Global well-posedness of the Benjamin-Ono equation in $H^1(\mathbb{R})$*, J. Hyperbolic Differ. Equ., 1, no. 1, 27-49, 2004.

Fractional polyanalyticity and applications

Paula Cerejeiras, Arran Fernandez, Uwe Kähler, Walaa Yasin
Eastern Mediterranean University, Northern Cyprus
arran.fernandez@emu.edu.tr

Starting from the definitions of fractional complex d-bar derivatives provided in [1], we consider the kernels of these operators (fractionally polyanalytic functions) and provide a complete characterisation of such functions. We further construct a fractional Cauchy-Kovalevskaya (CK) extension, obtaining fractionally complex polyanalytic functions by series expansions starting from real analytic functions, and investigate the nature of the Fischer decomposition / inner product in this fractional setting. If time allows, we will also discuss extensions of this theory from complex to hypercomplex settings, according to the fractional d-bar derivatives in quaternions and Clifford algebras [2,3].

References

[1] A. Fernandez, C. Bouzouina, *Fractionalisation of complex d-bar derivatives*, Complex Variables and Elliptic Equations, 66(3), pp. 437-475, 2021.

[2] A. Fernandez, C. Güder, W. Yasin, *Fractional powers of the quaternionic d-bar derivative*, Advances in Applied Clifford Algebras, 34, 2, 2024.

[3] W. Yasin, A. Fernandez, *Fractional powers of Clifford d-bar and radial derivatives*, Journal of Mathematical Analysis and Applications, 542(2), 128952, 2025.

Factorization of the time-fractional telegraph equation and solutions

Milton Ferreira
School of Technology and Management, Polytechnic University of Leiria, Portugal
milton.ferreira@ipleiria.pt

The time-fractional telegraph equation has emerged as an important model for describing wave propagation and transport phenomena in complex media with memory effects. In this talk, we present a factorization approach for a multi-dimensional time-fractional telegraph equation involving Hilfer fractional derivatives. Using the Dirac method together with a suitable triplet of Pauli matrices, the telegraph equation is decomposed into a coupled system of two-term time-fractional diffusion Dirac-type equations.
The analysis is carried out through an operational framework that combines the Fourier transform in the spatial variable with the Laplace transform in time. This approach leads to explicit solution representations and reveals new Fourier transform pairs connecting Fourier kernels expressed through bivariate Mittag-Leffler functions with Fox H-functions of two variables. These connections provide a powerful tool for obtaining closed-form solutions in both the Fourier-time and space-time domains.
Particular attention is devoted to the asymptotic behaviour of the solutions, highlighting the influence of the fractional parameters and the memory effects inherent to the model. Graphical illustrations are presented to support the theoretical findings and to provide further insight into the dynamics of the solutions. We also show that the factorization procedure is not unique: alternative triplets of Pauli matrices generate related systems and corresponding families of solutions. Finally, if time permits, we will outline a possible extension of the present approach to multi-order fractional equations.
This is a joint work with M.M Rodrigues and N. Vieira (CIDMA - University of Aveiro).

References

[1] M. Ferreira, M.M. Rodrigues, N. Vieira, *Factorization à la Dirac applied to the time-fractional telegraph equation*, Commun. Nonlinear Sci. Numer. Simul., Vol. 102, 109420, 2026.

Function theories in Cayley-Dickson algebras

This session highlights recent developments in function theories over Cayley–Dickson algebras, with an emphasis on non-associative structures. These include octonions, split-octonions, complex octonions, sedenions, general alternative and twisted $\mathbb{Z}_2$-graded algebras, the Okubo algebra, and related systems. Topics include – but are not limited to – Fourier transforms and wavelet theory in this context, non-associative analysis and geometry, octonionic functional analysis, and their specific applications to physics and non-commutative geometry.

Session organisers: Sören Kraußhar (Universität Erfurt, Germany), Helmuth R. Malonek (Universidade de Aveiro, Portugal), Guangbin Ren (University of Science and Technology of China).

Models and semi-models for slice regular functions on real division algebras

Amedeo Altavilla, Cinzia Bisi
`University of Bari Aldo Moro, Italy`
`amedeo.altavilla@uniba.it`

Slice regular functions on the real division algebras $\mathbb{H}$ and $\mathbb{O}$ display a characteristic zero geometry: besides real and spherical zeros, they may possess isolated nonreal zeros, whose position is strongly affected by the noncommutative, and in the octonionic case nonassociative, algebraic structure. Recent orbit-theoretic results show that natural equivalence classes of slice regular functions are governed by the trace, the symmetrization and the central divisor, namely by the triple

$$\big(\mathrm{Tr}(f), N(f), \mathrm{cdiv}(f)\big),$$

as established in the quaternionic and octonionic settings in [2,3]. In the quaternionic case, preserving only $\mathrm{Tr}(f)$ and $N(f)$ leads instead to conjugacy by slice semiregular functions [1].

Motivated by these equivalence theories, we study whether a slice regular function can be replaced by a representative whose isolated nonreal zeros are all aligned in one complex slice $\mathbb{C}_I$. We call such a representative a *model* when it remains in the strong-equivalence class of the original function. We give a criterion for the existence of one-slice-preserving models and prove that every purely vectorial slice regular function admits one. On the other hand, explicit quaternionic examples show that models do not exist in general: the

central divisor creates a genuine obstruction to alignment within a strong-equivalence class.
We therefore introduce the weaker notion of a *semi-model*. A semi-model is allowed to leave the strong-equivalence class, but it preserves the symmetrization and the complete real and spherical part of the zero set, including multiplicities, while placing every isolated nonreal zero in a single slice. Our main result states that every slice regular function f on a basic circular domain in $\mathbb{H}$ or $\mathbb{O}$, with $N(f) \not\equiv 0$, admits a semi-model. A key step is a two-direction reduction: up to strong equivalence, the vector part of an arbitrary function can be made to take values in the complexification of a fixed quaternionic subalgebra. The construction highlights the distinct roles of trace, symmetrization and central divisor, and clarifies the gap between holomorphic automorphism orbits and semiregular conjugacy.

References

[1] A. Altavilla, C. de Fabritiis, *Equivalence of slice semi-regular functions via Sylvester operators*, Linear Algebra Appl., 607:151-189, 2020.

[2] C. Bisi, J. Winkelmann, *Invariants and automorphisms for slice regular functions*, J. Noncommut. Geom., 20(1):325-359, 2026.

[3] C. Bisi, J. Winkelmann *Invariants and automorphisms for slice regular functions: the octonionic case*, Proc. Roy. Soc. Edinburgh Sect. A, to appear.

Dunkl approach to slice regular functions on real alternative $*$-algebras

Giulio Binosi, **Hendrik De Bie, Pan Lian, Alessandro Perotti**
University of Trento, Italy
binosi@altamatematica.it

We establish a connection between Dunkl and Slice analysis in the setting of Clifford algebras, by characterizing sets of slice functions through Dunkl operators. More precisely, we prove that a Clifford algebra-valued function is slice if and only if it belongs to the kernel of the Dunkl-spherical-Dirac operator, and that a slice function is slice regular if and only if it belongs to the kernel of the Dunkl–Cauchy–Riemann operator. In order for the correspondence to hold, the multiplicity function defining Dunkl operators must satisfy a condition depending only on the dimension of the algebra and is closely related to the Sce exponent. An analogous correspondence (with a different condition) also holds for monogenic functions.
Building on these results we use the inverse Dunkl intertwining operator to propose a new method for constructing monogenic functions from a given holomorphic function, in

the spirit of Fueter's theorem, but in a completely different and more efficient way. Time permitting, we will discuss how this construction extends to arbitrary real alternative $*$-algebras.

References

[1] G. Binosi, H. De Bie, L. Pan, *Dunkl approach to slice regular functions*, Annali di Matematica Pura ed Applicata, 204, 2333–2350, 2025.

[2] G. Binosi, A. Perotti, *Dunkl regularity over alternative $*$-algebras*, arXiv:2602.01907, 2026.

Slice regular functions on alternative $*$-algebras: Prescribing zeros and values on discrete sets and related extension problems

Cinzia Bisi, Joerg Winkelmann
Ferrara University, Italy
bsicnz@unife.it

Slice regular functions are a generalization of holomorphic functions where alternative real $*$-algebras are considered instead of the

field of complex numbers. For such functions we show that zero sets and function values on suitable subsets may be prescribed. As a consequence, we show that for any axially symmetric domain there exist slice regular functions which (due to the nature of its zero set) can not be extended to a larger such domain.

References

1 R. Ghiloni, A. Perotti, *Slice regular functions on real alternative algebras*, Adv. Math., 226(2):1662-1691, 2011.

2 G. Gentili, I. Vignozzi, *The Weierstrass factorization theorem for slice regular functions over the quaternions*, Ann. Global Anal. Geom., 40(4):435-466, 2011.

3 J. Prezelj, F, Vlacci, *Vanishing of quaternionic cohomology groups and applications*, Arxiv, 2024.

Reconstruction of a slice regular function from some of its real components

Cinzia Bisi, Antonio Carbone, Riccardo Ghiloni
University of Ferrara, Italy
antonio.carbone@unife.it

It is well known that if two holomorphic functions f and g, defined on a domain D of $\mathbb{C}$, have the same real part, then they coincide on D up to an additive constant. An analogous result holds true for slice regular functions of one quaternionic variable, namely, if two slice regular functions f and g, defined on a circular slice domain Ω of the quaternions $\mathbb{H}$, have the same real part, then they coincide on Ω up to an additive constant. Even if not explicitly present in the literature, the same holds true for slice regular functions defined on circular slice domains Ω of the octonions $\mathbb{O}$.
The purpose of this seminar is to extend (and adapt) these kind of results to more general classes of algebras, including, among many others, the Clifford algebras $\mathbb{R}_{p,q}$ and the split octonions $\mathbb{SO}$.

A multipoint Schwarz–Pick lemma for slice regular functions over the quaternions

Davide Cordella
Marche Polytechnic University, Italy
d.cordella@staff.univpm.it

The classical Schwarz-Pick Lemma controls the contraction of holomorphic self-maps of the unit disk with respect to the Poincaré metric and underlies Nevanlinna-Pick interpolation theory. Beardon and Minda, and later Baribeau, Rivard and Wegert, refined this picture through (iterated) hyperbolic difference quotients, obtaining sharp multipoint versions of the Schwarz-Pick Lemma.
In this talk we present a quaternionic counterpart of these results, in the setting of slice regular functions on the quaternionic unit ball $\mathbb{B}$, building on the one-point Schwarz-Pick Lemma of Bisi and Stoppato. Relying on basic features of slice regularity (the $*$-product, regular Möbius transformations and Blaschke products), we can introduce a regular hyperbolic difference quotient and derive a three-point Schwarz-Pick Lemma for slice regular self-maps of $\mathbb{B}$, which can be further generalized to more than three points. As consequences, one obtains quaternionic analogues of the classical Dieudonné and Goluzin distortion estimates for the hyperbolic derivative of self-maps fixing the origin.

Finally, restricting to real interpolation nodes (a restriction forced by the noncommutativity of $\mathbb{H}$), we describe an explicit algorithm, in the spirit of Schur and Nevanlinna, solving the n-point Nevanlinna-Pick interpolation problem for slice regular self-maps of $\mathbb{B}$: given real nodes $r_1, \ldots, r_n \in (-1, 1)$ and target values $s_1, \ldots, s_n \in \mathbb{B}$, we can construct quantities Q^ℓ_κ by iterating hyperbolic difference quotients and show that an interpolating function exists if and only if the final quantity has modulus at most one, with uniqueness, given by a regular Blaschke product, on the boundary case. We will also discuss the obstructions that appear when the nodes are not real.
This talk is based on joint work with Cinzia Bisi (Università di Ferrara).

References

[1] C. Bisi, D. Cordella, *Multipoint Schwarz-Pick lemma for the quaternionic case*, J. Geom. Anal., 36, 155, 2026.

[2] A.F. Beardon, D. Minda, *A multi-point Schwarz-Pick lemma*, J. Anal. Math., 92, 81-104, 2004.

[3] L. Baribeau, P. Rivard, E. Wegert, *On hyperbolic divided differences and the Nevanlinna-Pick problem*, Comput. Methods Funct. Theory, 9(2), 391-405, 2009.

[4] C. Bisi, C. Stoppato, *The Schwarz-Pick lemma for slice regular functions*, Indiana Univ. Math. J., 61(1), 297-317, 2012.

[5] D. Alpay, V. Bolotnikov, F. Colombo, I. Sabadini, *Self-mappings of the quaternionic unit ball: multiplier properties, the Schwarz-Pick inequality, and the Nevanlinna-Pick interpolation problem*, Indiana Univ. Math. J., 64(1), 151-180, 2015.

[6] G. Gentili, C. Stoppato, D.C. Struppa, *Regular functions of a quaternionic variable*, 2nd edn. Springer Monographs in Mathematics, Springer, Cham, 2022.

Decomposition of slice-regular functions

Chiara de Fabritiis
`Marche Polytechnic University, Italy`
`c.defabritiis@univpm.it`

Slice regular functions over quaternions were introduced by G. Gentili and D. Struppa in the seminal paper [1] which appeared on in 2006, exactly 20 years ago. Almost instantaneously the theory had broad developments in several directions, going from function theory on domains on the quaternions to functional calculus, from various generalizations to (non-commutative/non-associative) algebras to the study of (slice) regular functions in

several variables. It is almost impossible to trace all the different areas involved in these advancements, but the few references included ([2]-[6]) can help the reader to trace, at least partially, the growth of the discipline.
Our study deals with what happens in detail when the $*$-product of two slice-regular function on the quaternions is involved. Indeed, due to the skewness of the quaternions, in general the pointwise product of slice-regular functions is no more slice-regular, so that a different kind of product is needed in order to obtain an associative algebra of functions which turns out to be non abelian (a different form of the $*$ product was given in [7], where slice-regular commuting functions were detailedly portraited). A characterization of anticommuting functions will be presented in the talk, together with an analysis of the necessary (and/or) sufficient conditions for which a splitting property in summands which commute and anti-commute with a given function holds for slice-regular functions.

References

[1] G. Gentili, D. C. Struppa, *A new approach to Cullen-regular functions of a quaternionic variable*, Comptes Rendus. Mathématique, 342(10), pp. 741-744, 2006.

[2] G. Gentili, C. Stoppato, D. C. Struppa, *Regular Functions of a Quaternionic Variable (II edition).* Springer Cham, New York, 2022.

[3] F. Colombo, I. Sabadini, D. C. Struppa, *Noncommutative Functional Calculus Theory and Applications of Slice Hyperholomorphic Functions.* Springer, Basel, 2011.

[4] R. Ghiloni, A. Perotti, *Slice regular functions in several variables*, Math. Z. 302, 295-351, 2022.

[5] R. Ghiloni, C. Stoppato, *A unified theory of regular functions of a hypercomplex variable*, Bulletin des Sciences Mathématiques, Volume 209, 2026.

[6] R. Ghiloni, A. Perotti, C. Stoppato, *Division algebras of slice functions*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 150(4):2055-2082, 2020.

[7] A. Altavilla, C. de Fabritiis, *s-Regular Functions which Preserve a Complex Slice*, Annali di Matematica Pura e Applicata, Vol. 197, N. 4, p. 1269-1294, 2018.

Split-octonions, zero divisors, and applications to cryptography

Youssef El Haoui
Moulay Ismail University, Meknès, Morocco
y.elhaoui@umi.ac.ma

Split-octonions form an eight-dimensional non-associative and non-commutative algebra arising from the Cayley-Dickson construction. Unlike classical octonions, they are equipped with an indefinite quadratic form and admit zero divisors, leading to distinctive algebraic properties. Leveraging this structure, we propose a key-exchange protocol in which parties generate random non-invertible identifiers within the split-octonion algebra. The public communication identifier is constructed, while the exchanged subkeys are combined to produce a shared session key. Efficient algorithms are developed together with procedures for coefficient encoding and decoding. Numerical examples illustrate the feasibility of the proposed scheme.

Slice regularity, conformality and Riemann manifolds on quaternions and octonions

Graziano Gentili
University of Florence, Italy
graziano.gentili@unifi.it

This talk begins with a brief presentation of the fundamental features of (slice) regularity of functions over quaternions, alongside recent advancements in the field. Several applications of this theory will be indicated, including the one concerning the study of orthogonal complex structures (OCSs) on domains of $\mathbb{R}^4$.
Quaternionic and octonionic analogs of classical Riemann surfaces will be presented. The construction of these manifolds has distinct peculiarities; a re-examination of the classical approach to Riemann surfaces, which is primarily based on conformality, leads to the definition of slice conformal or slice isothermal parameterizations for quaternionic and octonionic Riemann manifolds. These classes of manifolds encompass slice regular quaternionic and octonionic curves, graphs of slice regular functions, 4- and 8-dimensional spheres, as well as helicoidal and catenoidal 4- and 8-dimensional manifolds. Additionally, appropriate Riemann manifolds provide a unified framework for defining quaternionic and octonionic logarithms and n-th root functions.
The presentation will conclude by indicating recent results concerning affine quaternionic curves and surfaces.

A comparison of two theories of regular functions of several octonionic variables

Cinzia. Bisi, Antonio Carbone, Riccardo Ghiloni
University of Trento, Italy
riccardo.ghiloni@unitn.it

We compare the theories of slice regular functions in several octonionic variables introduced in [3] and [4] (see also [1,2]). Our main results extend from octonions to all finite dimensional alternative $*$-algebras.

References

[1] X. Dou, G. Ren, I. Sabadini, *A representation formula for slice regular functions over slice-cones in several variables*, Ann. Mat. Pura Appl., (4) 202, no. 5, 2421-2446, 2023.

[2] X. Dou, G. Ren, I. Sabadini, T. Yang, *Weak slice regular functions on the n-dimensional quadratic cone of octonions*, J. Geom. Anal., 31, no. 11, 11312-11337, 2021.

[3] R. Ghiloni, A. Perotti, *Slice regular functions in several variables*, Math. Z., 302, no. 1, 295-351, 2022.

[4] G. Ren, T. Yang, *Slice regular functions of several octonionic variables*, Math. Methods Appl. Sci., 43, no. 9, 6031-6042, 2020.

An $\mathfrak{sl}_2$-Fischer decomposition for ultrahyperbolic operators with applications to complex octonion analysis

Heikki Orelma
Tampere Institute of Mathematics, Tampere, Finland
heikki.orelma@proton.me

In [1], the ultrahyperbolic complex operator

$$L = \Delta_x - \Delta_y + 2i \sum_{j=1}^{n} \partial_{x_j} \partial_{y_j} = 4 \sum_{j=1}^{n} \partial_{\overline{z}_j}^2,$$

where $z = x + iy \in \mathbb{C}^n$, was introduced as the canonical scalar operator associated with the octonionic Cauchy-Riemann operator in dimension $n = 8$. Owing to its ultrahyperbolic nature, the operator does not admit a standard Fischer decomposition. However, a Fischer-type decomposition can be obtained by considering the operator L together with its complex conjugate $\overline{L}$. These questions are investigated in this talk.
It is shown that the linear symmetry group of L is $O(n, \mathbb{C})$, while its affine symmetry group is $O(n, \mathbb{C}) \ltimes \mathbb{R}^{2n}$.
The $O(n, \mathbb{C})$-invariant functions are constructed using the basic invariants $r(z) = z^T z$ and $r(\overline{z}) = \overline{z}^T \overline{z}$; the corresponding radial equation is reduced to a Cauchy-Euler equation.
The main result is the Fischer-type decomposition for antiholomorphic polynomials:

$$\mathbb{C}[\overline{z}] = \bigoplus_{k=0}^{\infty} r(\overline{z})^k \ker L,$$

which further yields the decomposition of the full polynomial space

$$\mathbb{C}[z, \overline{z}] = \bigoplus_{i,j=0}^{\infty} r(z)^i r(\overline{z})^j \left(\ker \overline{L} \otimes \ker L\right).$$

The decomposition is based on an $\mathfrak{sl}_2$-structure generated by L, r, and the Euler operator.
Finally, the restriction of the decomposition to the null cone

$$\mathcal{C} = \{z \in \mathbb{C}^n \mid r(z) = 0\}$$

is studied. It is proved that the space of biharmonic polynomials $\ker \overline{L} \otimes \ker L$ can be identified with the space of polynomial functions on the null cone, leading to a Cauchy-Kovalevskaya-type extension theorem.

References

[1] R.S. Kraußhar, H. Orelma, *Towards a Function Theory of Complexified Octonions*, Ann Univ Ferrara 72, 83, 2026.

Fueter trees for Dunkl-regular functions over alternative $*$-algebras

Alessandro Perotti
University of Trento, Italy
alessandro.perotti@unitn.it

We present a general Fueter Theorem over real alternative $*$-algebras. We show that a suitable power of the Laplacian maps *Dunkl-regular functions* [2] to Dunkl monogenic functions [3] with axial symmetries. Recently [1,2] it was shown that monogenicity and slice-regularity, as well as several other hypercomplex function theories, can be characterized by means of Dunkl operators over hypercomplex subspaces of alternative $*$-algebras. This result allows to apply results of Dunkl theory to hypercomplex analysis. In the present talk we use properties of the Dunkl Laplacian to obtain a general Fueter Theorem. Using the embedding of hypercomplex function theories in the class of Dunkl monogenic functions, we subsume several Fueter-type results known in the literature and obtain the most general form for the action of the Laplacian on function spaces over hypercomplex subspaces. We show that Fueter Theorems are in a one-to-one correspondence with a class of graphs, the *Fueter trees*, that describe the interactions between Dunkl-regular function spaces and the relation with the iterated Laplacian. We obtain that the number of non-equivalent Fueter trees on a hypercomplex space of dimension $n+1$ is equal to the number of partitions in odd parts of the integer n.

References

[1] G. Binosi, H. De Bie, P. Lian, *Dunkl approach to slice regular functions*, Ann. Mat. Pura Appl., (4), 2025.

[2] G. Binosi, A. Perotti, *Dunkl regularity over alternative $*$-algebras*, 2025.

[3] P. Cerejeiras, U. Kähler, G. Ren, *Clifford analysis for finite reflection groups*, Complex Var. Elliptic Equ., 51(5-6):487-495, 2006.

Quaternionic and octonionic Cartan coverings and applications

Jasna Prezelj, Fabio Vlacci
University of Ljubljana, Slovenia
jasna.prezelj@fmf.uni-lj.si

We present the topological foundations for the solvability of multiplicative Cousin problems formulated on an axially symmetric domain $\Omega \subset \mathbb{H}, \mathbb{O}$. In particular, we provide a geometric construction of quaternionic Cartan coverings, which are generalizations of (complex) Cartan coverings. Because of the requirements of symmetry inherent to the domains of definition of quaternionic regular functions, the existence of quaternionic Cartan coverings of Ω is not a consequence of the existence of complex Cartan coverings; for the latter, there are no requirements for the symmetries with respect to the real axis. Due to the real axis's special role, the covering restricted to $\Omega \cap \mathbb{R}$ must have additional properties. All these required properties were achieved by starting from a particular symmetric tiling of the symmetric set $\Omega \cap (\mathbb{R} + i\mathbb{R})$. Finally, we apply these results to prove the vanishing of 'antisymmetric' cohomology groups of planar symmetric domains for $n \geq 2$.

References

[1] J. Prezelj, F. Vlacci, *Quaternionic Cartan Coverings and Applications*, J Geom Anal, 35, 80, 2025.

Generalized partial-slice monogenic functions in the octonionic case

Irene Sabadini, Zhenghua Xu
Polytechnic University of Milan, Italy
milton.ferreira@ipleiria.pt

n the paper [1] we introduced the concept of generalized partial-slice monogenic (or regular) function was introduced over Clifford algebras. To study can be extended from the associative case of Clifford algebras to non-associative alternative algebras, such as octonions. The new class of functions encompasses the regular functions and slice regular functions over octonions, indeed both appear in the theory as special cases. In the non-associative setting of octonions, we developed in [2] some fundamental properties such as identity theorem, Representation Formula, Cauchy (and Cauchy-Pompeiu) integral formula, maximum modulus principle, Fueter polynomials, Taylor series expansion. From the arguments in the proofs it is clear that the results hold more in general in real alternative algebras equipped with a notion of conjugation.

References

[1] Z. Xu, I. Sabadini, *Generalized partial-slice monogenic functions*, Trans. Amer. Math. Soc., 378, no. 2, 851-883, 2025.

[2] Z. Xu, I. Sabadini, *Generalized partial-slice monogenic functions: the octonionic case*, Trans. Amer. Math. Soc., 379, no. 5, 3257–3291, 2026.

Quaternionic Carleson measures

Giulia Sarfatti
Marche Polytechnic University, Italy
g.sarfatti@univpm.it

Carleson measures are a fundamental object in complex analysis, originally introduced by Carleson in his solution of the celebrated Corona Problem. They play a key role in several areas of complex analysis and operator theory. For example, they provide characterizations of the boundedness of Toeplitz, Hankel, and composition operators, and naturally arise in interpolation theory. Moreover, the subclass of vanishing Carleson measures allows one to characterize the compactness of the previously mentioned operators.
Slice regularity is a notion of holomorphicity for quaternionic functions, introduced by Gentili and Struppa in 2006, which has given rise to a rich and rapidly developing theory. In recent years, function spaces in this setting have been extensively investigated by several authors from a variety of perspectives.
In this talk, I will present a general construction that associates with a Banach space of holomorphic functions a quaternionic Banach space of slice regular functions, called its *quaternionic lift*. This framework includes many of the most relevant quaternionic Banach spaces of slice regular functions appearing in the literature.
I will then characterize Carleson and vanishing Carleson measures for these quaternionic Banach spaces in terms of the corresponding notions for the underlying holomorphic function spaces. This offers a unified approach to a problem that has so far been addressed only on a case-by-case basis.
This talk is based on a joint work with Nikolaos Chalmoukis.

References

[1] N. Chalmoukis, G. Sarfatti, *Quaternionic Carleson measures*, ArXiv Preprint.

On the first-order theory of octonions

Enrico Savi
University of Angers, France
enrico.savi@univ-angers.fr

In complex and real algebraic geometry, the question on how projections of algebraic sets behave is completely understood by the celebrated Chevalley's theorem and Tarski-Seidenberg principle, respectively. In general, the latter geometric problem has a very clear interpretation in model theory as the property of a theory to admit quantifier elimination with respect to an appropriate language. In turn, this interpretation and similar properties of model theoretical nature prove to be very useful for applications in algebra and geometry. The aim of this talk is to investigate the octonionic version of the latter problem and its consequences developed by the author in [1].
Denote by $L = \{+, -, *, 0, 1\}$ the language of rings. We characterize the models of the (first-order) L-theory of octonions as octonionic algebras $\mathbb{O}_R$ over a real closed field R and we prove some fundamental properties of the theory, such as completeness and model completeness. We observe that the theory of octonions does not have quantifier elimination with respect to the language L: additional symbols must be added to achieve this property. Then, we focus on the consequences in algebraic geometry over $\mathbb{O}_R$. We introduce a suitable definition of the Zariski topology of $\mathbb{O}_R^n$ and we describe the first properties of algebraic subsets of $\mathbb{O}_R^n$ we introduced, such as the real dimension of the zero set of an ordered polynomial and the topological types of octonionic algebraic sets.

References

[1] E. Savi, *On the first-order theories of quaternions and octonions*, Annali di Matematica Pura e Applicata, 2026.

Unified hypercomplex function theory and Fueter-Sce phenomenon

Riccardo Ghiloni[1], Caterina Stoppato[2]
University of Florence, Italy
caterina.stoppato@unifi.it

Recent work, [14,15], unified into a single theory several hypercomplex theories in one variable introduced and studied over the last century: the quaternionic theories of Fueter [7,8] and of Gentili-Struppa [9,10,11]; the octonionic ones, due to Dentoni-Sce [5] and to Gentili-Struppa [12]; the celebrated theory of Clifford monogenic functions [1,2,17] and the slice-monogenic theory of Colombo-Sabadini-Struppa [3,4]. In the Clifford case, related independent works are [19,20]. Following the steps of [13], the setting of [14,15] are general alternative $*$-algebras. In this generalized setting, [16] unifies Fueter's theorem [7], Sce's theorem [18] and the more recent results of Xu and Sabadini [21] into a single statement and generalizes them to associative $*$-algebras. Multi-axial symmetry, in a sense introduced by Eelbode [6], is produced. Additionally, but more surprisingly, [16] shows that the phenomenon studied by Fueter, Sce, Xu and Sabadini is just the last step in a multi-step process.

References

[1] F. Brackx, R. Delanghe, F. Sommen, *Clifford analysis.* Pitman, Boston, MA, 1982.

[2] J. Cnops, H. Malonek, *An introduction to Clifford analysis*, Textos de Matemática. Série B [Texts in Mathematics. Series B], 7. Universidade de Coimbra Departamento de Matemática, Coimbra, 1995.

[3] F. Colombo, I. Sabadini, D. C. Struppa, *Slice monogenic functions*, Israel J. Math., 171:385-403, 2009.

[4] F. Colombo, I. Sabadini, D. C. Struppa, *Noncommutative functional calculus. Theory and applications of slice hyperholomorphic functions*, volume 289 of Progress in Mathematics, Birkhäuser/Springer Basel AG, Basel, 2011.

[5] P. Dentoni, M. Sce, *Funzioni regolari nell'algebra di Cayley*, Rend. Sem. Mat. Univ. Padova, 50:251-267, 1973.

[6] D. Eelbode, *The biaxial Fueter theorem*, Israel J. Math., 201(1):233-245, 2014.

[7] R. Fueter, *Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen Variablen*, Comment. Math. Helv., 7(1):307-330, 1934.

[8] R. Fueter, *Über die analytische Darstellung der regulären Funktionen einer Quaternionenvariablen*, Comment. Math. Helv., 8(1):371-378, 1935.

[9] G. Gentili, C. Stoppato, D. C. Struppa, *Regular functions of a quaternionic variable.* Springer Monographs in Mathematics. Springer, Cham, 2022.

[10] G. Gentili, D. C. Struppa, *A new approach to Cullen-regular functions of a quaternionic variable*, C. R. Math. Acad. Sci. Paris, 342(10):741-744, 2006.

[11] G. Gentili, D. C. Struppa, *A new theory of regular functions of a quaternionic variable*, Adv. Math., 216(1):279-301, 2007.

[12] G. Gentili, D. C. Struppa, *Regular functions on the space of Cayley numbers*, Rocky Mountain J. Math., 40(1):225-241, 2010.

[13] R. Ghiloni, A. Perotti, *Slice regular functions on real alternative algebras*, Adv. Math., 226(2):1662-1691, 2011.

[14] R. Ghiloni, C. Stoppato, *A unified notion of regularity in one hypercomplex variable*, J. Geom. Phys., 202:Paper No. 105219, 2024.

[15] R. Ghiloni, C. Stoppato, *A unified theory of regular functions of a hypercomplex variable*, Bull. Sci. Math., 209:Paper No. 103794, 2026.

[16] R. Ghiloni, C. Stoppato, *A manifold Fueter-Sce phenomenon in one hypercomplex variable*, J. Math. Anal. Appl., Paper No. 130903, 2026.

[17] K. Gürlebeck, K. Habetha, W. Sprößig, *Holomorphic functions in the plane and n-dimensional space*. Birkhäuser Verlag, Basel, 2008.

[18] M. Sce. *Osservazioni sulle serie di potenze nei moduli quadratici*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8), 23:220-225, 1957.

[19] Z. Xu, I. Sabadini, *Generalized partial-slice monogenic functions: a synthesis of two function theories*, Adv. Appl. Clifford Algebras, 34(2):Paper No. 10, 14, 2024.

[20] Z. Xu, I. Sabadini, *Generalized partial-slice monogenic functions*, Trans. Amer. Math. Soc., 378(2):851-883, 2025.

[21] Z. Xu, I. Sabadini, *On the Fueter-Sce theorem for generalized partial-slice monogenic functions*, Ann. Mat. Pura Appl. (4), 204(2):835-857, 2025.

Quasi-polar algebras and quasi-Clifford systems

Vladimir G. Tkachev
Linköping University, Sweden
vladimir.tkatjev@liu.se

Symmetric Clifford systems are collections of symmetric operators satisfying Clifford-type relations. They give rise to highly structured polynomial maps with remarkable properties, which play a role in the study of harmonic and minimal immersions. They provide a natural source of polynomial maps with special spectral and symmetry properties with important applications in geometry (e.g. isoparametric foliations) and, more recently, in nonassociative algebraic constructions. In algebra, they lead to structured nonassociative constructions, including certain classes of Hsiang algebras and related commutative algebras with constrained spectra.
The present talk has two principal aims. The first is to recall certain facts about symmetric Clifford systems, introduced in [1] and further developed in the context of polar

algebras in [5], which have recently appeared in a variety of purely algebraic [7] and differential-geometrical contexts [4], [5], including radial ∞-Hsiang algebras [2], [3]. The second is to announce a general construction based on the notion of a quasi-Clifford system, which generalizes the classical concept of a Clifford system, along with several recent applications. The talk is based on joint work with D. J. Fox. (Universidad Politécnica de Madrid).

References

[1] D. Ferus, H. Karcher, H. F. Münzner, *Cliffordalgebren und neue isoparametrische Hyperflächen*, Math. Z. 177, no. 4, 479–502, 1981.

[2] D. J. F. Fox, V. G. Tkachev, *Algebraic constructions of cubic minimal cones*, Pure Appl. Funct. Anal., 10(3), 2026.

[3] D. J. F. Fox, V. G. Tkachev, *Classification of ∞-Hsiang algebras of submaximal Peirce complexity*, Preprint, 92 p., 2026.

[4] Y. Lü, P. Wang, Z. Xie, *Classification of minimal immersions of conformally flat 3-tori and 4-tori into spheres by the first eigenfunctions*, Math. Ann., vol. 390, 2, 2235-2280, 2024.

[5] N. Nadirashvili, V. Tkachev, S. Vlăduţ, *Nonlinear elliptic equations and nonassociative algebras*, Math. Surv. Monographs, vol. 200, AMS, Providence, 2014.

[6] C. Qian, Z. Tang, W. Yan, *Clifford systems, harmonic maps and metrics with non-negative curvature*, Pacific J. Math., vol. 320, 2, 391-424, 2022.

[7] M. Parton, P. Piccinni, V. Vuletescu, *Clifford systems in octonionic geometry*, Rend. Sem. Mat. Univ. Pol. Torino, vol. 74, 1, 269–290, 2016.

Recent developments in the theory of algebraic sets for slice-regular quaternionic polynomials

Anna Gori, Giulia Sarfatti, Fabio Vlacci
University of Trieste, Italy
fvlacci@units.it

In this talk we'll shortly present some recent results obtained about the development of a theory of algebraic sets in the framework of slice-regular quaternionic polynomials. The project, which has been developed in the last two years, aims at extending some of the classical notions of (Complex) Algebraic Geometry in the quaternionic setting. At the moment we have introduced some new tools to better understand the geometric and algebraic structures underlying basic ideas which are promising to describe, in particular, intrinsic symmetries of the common zero loci of quaternionic slice-regular polynomials in several quaternionic variables.

References

[1] A. Gori, G. Sarfatti, F. Vlacci, *Zero sets and Nullstellensatz type theorems for slice regular quaternionic polynomials*, Linear Algebra Appl. 685, p. 162-181, 2024.

[2] A. Gori, G. Sarfatti, F. Vlacci, *A Strong Version of the Hilbert Nullstellensatz for slice regular polynomials in several quaternionic variables*, J. Algebra, 687, p. 269-291, 2026.

Generalizations of complex analysis: Clifford, hypercomplex analysis and beyond

Clifford analysis and more in general hypercomplex analysis can be regarded as an extension of complex analysis to higher dimensions. The field has attracted attention in several areas, as it provides an elegant way of dealing with problems of mathematical physics in higher dimensions, in operator theory, analysis of PDEs, spectral theory based on the S-spectrum, reproducing kernel methods and Schur analysis, among others. This session aims to present recent advances in the field, mostly for functions with values in associative algebras, as well as its various applications in related areas.

Session organisers: Swanhild Bernstein (TU Bergakademie Freiberg, Germany), Paula Cerejeiras (Universidade de Aveiro, Portugal), Fabrizio Colombo (Politecnico di Milano, Italy), Irene Sabadini (Politecnico di Milano, Italy).

Geometric singularities and Hodge theory

Lashi Bandara
Deakin University, Burwood, Australia
lashi.bandara@deakin.edu.au

A seminal result by Georges de Rham in 1931 was to establish an isomorphism between the singular cohomology and the de Rham cohomology on compact boundaryless manifolds. Given a smooth metric on such a manifold, a related result asserts that the space of harmonic forms is isomorphic to this cohomology. Beyond the benefit of introducing operator methods to the study of topology, this can be interpreted as prescribing a geometry to calculate the cohomology, making it possible to potentially choose an optimal metric for this task. However, there is no reason that such a metric needs to be smooth. Motivated by this, along with other considerations, we revisit Hodge theory in the context of singular metrics called Rough Riemannian metrics.
Utilising the seminal perspective developed by Keith-McIntosh-Rosén, a perturbative approach is taken to show that the kernels of the induced Hodge-Dirac operators remain isomorphic under uniform changes of metric. This has the added benefit of allowing us to consider non-compact manifolds as well as manifolds with boundary. These results are obtained as a special case of a more general context featuring nilpotent differential operators supported on a vector bundle equipped with a Rough Hermitian metric.

Q-Clifford analysis in quantum space

Swanhild Bernstein, Martha Zimmermann
TU Bergakademie Freiberg, Germany
swanhild.bernstein@math.tu-freiberg.de

A real Clifford algebra is typically generated by n elements that satisfy the relation

$$e_i e_j + e_j e_i = -2\delta_{i,j}.$$

In the case of $2n$ real generating elements, it is possible to construct a so-called Witt basis, which fulfills a Grassmann relation, the duality relation, and isotropy conditions. This basis can be identified with fermionic raising and lowering operators that adhere to the canonical anticommutation relations [3]. We will demonstrate how different settings in the Quantum space, i.e., q-commuting variables, give rise to different settings for a real-valued q-Clifford algebras, where we use different algebras for the Dirac operator and the vector variable. We extend results from [2]. A complex-valued q-Clifford algebra (generalized Witt basis). is constructed from the twisted canonical anticommutation relations [1] based on the pseudogroup $SU_q(n)$ where $0 < q < 1$. This approach is quite different then in the real-valued case. We will discuss related topics such as Spin groups.

References

[1] W. Pusz, *Twisted Canonical Anticommutation Relations*, Reports on Mathematical Physics, 27(3):349-360, 1989.

[2] S. Bernstein, M.L. Zimmermann, B. Schneider, *The q-Dirac Operator on Quantum Euclidean Space.* In: Alpay, D., Lewkowicz, I., Vajiac, A., Vajiac, M. (eds), Schur Analysis and Applications to Hypercomplex Analysis, Neural Networks, and Linear Systems. WSAA 2024. Operator Theory: Advances and Applications, vol 308, Birkhäuser, Cham, 2026.

[3] P. Woit. *Quantum Theory, Groups and Representations.* Springer, 2017.

The Dirac operator in projective geometric algebra

Danail Brezov
University of Architecture, Civil Engineering and Geodesy, Bulgaria
danail.brezov@gmail.com

In this paper we consider the Dirac operator in Projective Geometric Algebra (PGA) associated with three-dimensional (affine) Euclidean space and derive analogues to the Maxwell's equations viewed as a condition for analyticity generalising the familiar Cauchy-Riemann condition to bi-quaternions. Both the point-based and its dual, the plane-based version of PGA, are studied, and the action of the operator on flats of different grades is examined, relating the results to familiar problems in physics, geometry and engineering. Specific examples are provided for illustration and worked out in detail. The links to Conformal Geometric Algebra (CGA), in which PGA is embedded, are also discussed.

References

[1] R. Delanghe, *Clifford Analysis: History and Perspective*, Comput. Methods Funct. Theory, 1, 107-153, 2001.

[2] J. Hrdina, A. Návrat, P. Vašík, L. Dorst, *Projective Geometric Algebra as a Subalgebra of Conformal Geometric Algebra*, Adv. Appl. Clifford Algebras, 31(18), 2021.

[3] L. Dorst, S. De Keninck, *A Guided Tour to the Plane-Based Geometric Algebra PGA*, 2022.

Fine structures on the S-spectrum

Fabrizio Colombo
Polytechnic University of Milan, Italy
fabrizio.colombo@polimi.it

The spectral theory on the S-spectrum originated to provide a precise mathematical foundation for quaternionic quantum mechanics and a spectral theory for linear operators in vector analysis. It has proven significantly more general than expected, extending to Clifford operators and showing deep connections with the monogenic spectral theory developed by A. McIntosh and collaborators. In recent years, combining slice hyperholomorphic functions with the Fueter-Sce mapping theorem we have enlarge the classes of functions to which this theory works. This generalization defines the so called *fine structures on the S-spectrum*, and is based on classes of function with integral representations from which we define the associated functional calculi.

Resolvent equations on the fine structures

João Marques da Costa
Polytechnic University of Milan, Italy
joao.marquesdacosta@polimi.it

The fine structures of the S-spectrum constitute a new research area including a class of functional calculi based on the S-spectrum and on integral transforms arising from the Fueter–Sce mapping theorem and the Cauchy formula for slice hyperholomorphic functions. This approach provides functional calculi that include several previously known calculi, namely the poliharmonic functional calculus. The resolvent operators in this setting do not arise directly from a Cauchy kernel, but rather from suitable manipulations of it. For this reason the corresponding resolvent equations differ substantially from those associated with the classical Cauchy kernel. In this talk, we present the resolvent equations in dimension five, as well as the corresponding product rules.
This is joint work with F.Colombo and A. de Martino.

References

[1] F. Colombo, A. De Martino, J.M. Da Costa, *The resolvent equations for the Harmonic and bi-Harmonic functional calculi in dimension five*, arXiv:2603.15349, 2026.

On the harmonic generalized Cauchy-Kovalevskaya extension and its connection with the Fueter-Sce theorem

Antonino De Martino, Ali Guzmán Adán
Polytechnic University of Milan, Italy
antonino.demartino@polimi.it

In this talk we establish a generalized Cauchy-Kovalevskaya extension for axially harmonic functions. We demonstrate that the result can be expressed as a power series involving Bessel-type functions of specific differential operators acting on two initial functions. Additionally, we analyze the decomposition of the harmonic CK extension in terms of integrals over the sphere $\mathbb{S}^{m-1}$ involving functions of plane wave type.
Another key goal of this talk is to explore the relationship between the harmonic Cauchy-Kovalevskaya extension and the Fueter-Sce theorem. The Fueter-Sce theorem outlines a two-step process for constructing axially monogenic functions in $\mathbb{R}^{m+1}$ starting from holomorphic functions in one complex variable. The first step generates the class of slice monogenic functions, while the second step produces axially monogenic functions

by applying the pointwise differential operator $\Delta_{\mathbb{R}^{m+1}}^{\frac{m-1}{2}}$ with m being odd, known as the Fueter-Sce map, to a slice monogenic function.
By suitably factorizing the Fueter-Sce map, we introduce the set of axially harmonic functions, which serves as an intermediate class between slice monogenic and axially monogenic functions. In this talk, we establish a connection between the harmonic CK extension and the factorization of the Fueter-Sce map.

Octonionic Riesz-Dunford functional calculus

Qinghai Huo, Guangbin Ren, Irene Sabadini, Zhenghua Xu
Hefei University of Technology, China
hqh86@mail.ustc.edu.cn

The Riesz-Dunford functional calculus over the algebra of octonions, has long been an open problem due to the nonassociativity of octonions. Two core obstacles hinder its development: first, the generalization of the resolvent operator series identity produces unexpected associator terms that invalidate standard expansions; second, the nonassociativity spoils the analyticity of the resolvent operator, a key property for defining a functional calculus via Cauchy integrals. In this talk, we introduce the study of the Riesz-Dunford functional calculus for bounded power-associative para-linear operators in Banach octonionic bimodules. To address the above issues, we introduce several pivotal concepts: power-associative operators (to eliminate the unwanted associator terms and recover valid resolvent series expansions), the notions of regular inverse of $R_s - T$ for $s \in \emptyset$ (which serve as the octonionic versions of the resolvent operator), $\mathbb{C}_J$-extendable power-associative operators, and $\mathbb{C}_J$-liftable power-associative operators (to characterize the slice regularity of the resolvent operators). Based on these notions, we define two types of octonionic spectra: the pull-back spectrum $\sigma^*(T)$ and the push-forward spectrum $\sigma_*(T)$. These give rise to the left and right slice regular functional calculi of bounded power-associative para-linear operators, respectively.

References

[1] F. Colombo, I. Sabadini, D. C. Struppa, *Noncommutative functional calculus*, volume 289 of Progress in Mathematics. Birkhäuser/Springer Basel AG, Basel, 2011.

[2] Q. Huo and G. Ren, *Non-associative Categories of Octonionic Bimodules*, Commun. Math. Stat., 13(2):303-369, 2025.

Uwe Kähler
University of Aveiro, Portugal
ukaehler@ua.pt

Classic hypercomplex analysis is intimately linked with elliptic operators, such as the Laplacian or the Dirac operator, and positive quadratic forms. But there are many applications like the inversion of the crystallographic X-ray transform or the study of null-solutions of the ultrahyperbolic Dirac operator which are closely connected with indefinite quadratic forms. Furthermore, they also appear in statistics different from the standard Fermi or Bose-Einstein statistics. Among other things this is due to the possibility of the underlying Clifford algebra to have a signature and, therefore, to be linked to Pontryagin modules instead of Hilbert modules. Although in the majority of papers Hilbert modules are being used in this context they are not the right choice as function spaces since they do not reflect the induced geometry. In this talk we are going to show that Clifford-Krein modules are naturally appearing in this context. We take a particular look into the case of Clifford-Krein modules with reproducing kernels and show several applications where such spaces appear.

Some properties of biquaternions seen as a bicomplex module

M. Elena Luna-Elizarrarás
Holon Institute of Technology, Israel
lunae@hit.ac.il

It is well known that the ring of biquaternions $\mathbb{H}(\mathbb{C})$ is a real linear space of dimension eight. But it is also a four dimensional complex linear space, a quaternionic module, a bicomplex module and a hyperbolic module. In this talk we will analyze how the combination of the above structures reveal new aspects on the analytical properties of functions defined on $\mathbb{H}(\mathbb{C})$.

On recent applications of discrete Clifford analysis in quantum mechanics

Helmuth R. Malonek
University of Aveiro, Portugal
hrmalon@ua.pt

Following T. Gowers, we regard discrete Clifford analysis as a branch of mathematical analysis that yields results with an analytic flavor while also being related to discrete structures. The aim of this contribution is to present the analytical background for applications of results from discrete Clifford analysis to quantum-mechanical problems, first identified in 2021 in [1]. Years earlier, and based exclusively on the algebraic and combinatorial properties of generalized hypercomplex Appell polynomials, these results had already been obtained in purely combinatorial studies in [2]. Interestingly, they are expressed in the form of triangular and trapezoidal number structures, as described in [3]. We conclude by discussing similar combinatorial structures that were first mentioned in 2025 in [4] in the context of the scattering theory of Dirac delta potentials.

References

[1] P. Árendás, C.G. Császár, *Comment on "Wigner numbers"*, J. Chem. Phys. 154, 087101, 2021.

[2] I. Cação, M.I. Falcão, H.R. Malonek, G. Tomaz, *Combinatorial identities associated with a multidimensional polynomial sequence*, J. Integer Sequences 21, 18.7.4, 2018.

[3] I. Cação, M.I. Falcão, H.R. Malonek, G. Tomaz, *Pascal Trapezoids as Wigner Numbers and Some Combinatorial Properties.* In: Gervasi, O., et al. Computational Science and Its Applications – ICCSA 2025, Lecture Notes in Computer Science, Vol. 15887. Springer, Cham., 2026

[4] J. Figueroa, I. Gonzalez, D. Salinas-Arizmendi, *A novel transfer matrix framework for multiple Dirac delta potentials*, Physics Letters A, 555, Paper No. 130785, 2025.

Fractional powers of bisectorial operators and applications to the nonlocal Fourier's law

Francesco Mantovani
Polytechnic University of Milan, Italy
francesco.mantovani@polimi.it

In this talk, we introduce a notion of fractional powers of bisectorial operators in the Clifford setting, with particular emphasis on gradient operators with non-constant coefficients. The construction is based on the spectral theory of the S-spectrum and addresses a fundamental difficulty that distinguishes bisectorial operators from the classical sectorial case: the fractional power function is not analytic on the negative real axis and therefore cannot be directly treated within the standard functional calculus framework. To overcome this obstacle, we propose a new definition of fractional powers tailored to the bisectorial setting. We then apply the resulting functional calculus for vector operators to gradient operators, establishing a rigorous framework for defining and studying their fractional powers. As a main application, we show that these operators provide a mathematically solid foundation for nonlocal Fourier laws arising in models of heat propagation, thereby connecting the abstract operator-theoretic construction with nonlocal diffusion phenomena.
Joint work with Fabrizio Colombo and Peter Schlosser.

References

[1] F. Colombo, F. Mantovani, P. Schlosser, *Nonlocal Fourier laws for heat propagation via fractional powers of vector operators*, arXiv:2604.1321, 2026.

Solutions to time-harmonic Maxwell equations via transmutation theory

Pablo Enrique Moreira-Galván
Anáhuac Querétaro, Mexico
pablo.moreira@anahuac.mx

The use of complex quaternions provides a natural framework for representing a wide range of physical systems, including Maxwell and Beltrami fields. By extending the space of complex vector-valued functions, this approach allows for a unified treatment of

scalar and vector components, analogous to the role of complex analysis in real-variable problems. Consider the time-harmonic Maxwell system who has the form:

$$\text{curl } \vec{H} = -\mathbf{i}\omega\varepsilon\vec{E}, \qquad \text{curl } \vec{E} = \mathbf{i}\omega\mu\vec{H}, \tag{8}$$

$$\text{div } \vec{E} = 0, \qquad \text{div } \vec{H} = 0. \tag{9}$$

We transform the system (8)-(9) considering the vectorial functions $\vec{u} = (\sqrt{\epsilon}\vec{E} + \mathbf{i}\sqrt{\mu}\vec{H})$, $\vec{v} = (\sqrt{\epsilon}\vec{E} - \mathbf{i}\sqrt{\mu}\vec{H})$ and the number $\lambda = \omega\sqrt{\mu\epsilon}$ then the system is equivalent to solve:

$$\text{curl } \vec{u} = \lambda\vec{u}, \quad \text{curl } \vec{v} = -\lambda\vec{v}. \tag{10}$$

That means $\vec{u} \in \text{Ker}(D - \lambda)$ and $\vec{v} \in \text{Ker}(D + \lambda)$ where D is the Moisil-Theorodescu differential operator.
We construct operators $\mathbf{T}^{\pm}$ such that any function in $\text{Ker}(D \pm \lambda)$ admits the decomposition

$$\mathcal{P}^{+}\mathbf{T}^{+}[u_1] + \mathcal{P}^{-}\mathbf{T}^{-}[u_2] \in \text{Ker}(D + \lambda), \tag{11}$$

$$\mathcal{P}^{-}\mathbf{T}^{+}[u_3] + \mathcal{P}^{+}\mathbf{T}^{-}[u_4] \in \text{Ker}(D - \lambda), \tag{12}$$

where u_i are monogenic functions and $\mathcal{P}^{\pm}$ are projection operators.
This yields a constructive representation of solutions to the Maxwell system via the image of the operator $\mathbf{T}^{\pm}$. In particular, any electromagnetic field $(\vec{E}, \vec{H})$ defined in $B_R(0)$ admits a series expansion in terms of $\mathbf{T}^{\pm}[P^{l,m}]$, where $\{P^{l,m}\}$ is a complete system of monogenic polynomials.

A variation of the Fueter theorem and the generalized polyanalytic Cauchy-Kovalevskaya extension of order 2

Stefano Pinton, Antonino De Martino
Polytechnic University of Milan, Italy
stefano.pinton@polimi.it

In this talk, I will first present a generalized Cauchy–Kovalevskaya (GCK) extension theorem for axially polyanalytic functions of order 2. I will show that this extension can be expressed as a power series involving suitable differential operators acting on two initial functions. I will also describe a representation of the GCK extension in terms of integrals over the sphere $\mathbb{S}^2$, involving plane-wave-type functions.
Next, I will discuss the relationship between the GCK extension for polyanalytic functions of order 2 and the Fueter theorem, one of the fundamental results in hypercomplex analysis. The Fueter theorem can be viewed as a two-step procedure. In the first step, a suitable operator transforms holomorphic functions of one complex variable into slice hyperholomorphic functions. In the second step, the application of the Laplace operator

in four real variables—the so-called Fueter map—produces axially monogenic functions, namely functions belonging to the kernel of the Fueter operator.
A suitable factorization of the Fueter map in terms of the Fueter operator and its conjugate naturally leads to two intermediate classes of functions lying between slice hyperholomorphic and axially monogenic functions: axially harmonic functions and axially polyanalytic functions of order 2.
Finally, I will discuss the invertibility of the factorized Fueter map for harmonic and polyanalytic functions of order 2 on suitable domains. I will also derive integral representation formulas for the inverses of these factorizations. These representations are based on the Cauchy integral formula for polyanalytic functions of order 2 and on the Poisson integral formula for harmonic functions.

Reduced quaternionic Cauchy-like integrals

Isidro Paulino-Basurto, Juan Bory-Reyes, José Oscar González-Cervantes, Baruch Schneider
University of Ostrava, Czech Republic
baruch.schneider@osu.cz

The primary aim of the present talk is to derive an analogue of the Sokhotski-Plemelj formulas for the reduced quaternionic Cauchy-like integral in the context of $\mathcal{A}$-valued functions on domains of $\mathbb{R}^3$ with Ahlfors-regular boundary, which is considered distinct from that of quaternionic analysis, despite their natural similarities.
Our second task is to establish a Poincaré-Bertrand formula for interchanging the order of integration of two repeated singular reduced quaternionic Cauchy-like integral, drawing on arguments involving quaternionic analysis.
And lastly, a Geza Freud type theorem, and also some local properties of the singular reduced quaternionic Cauchy-like integral are proved.

Classifying slice-regular polynomials via group actions on the twistor space

Chunlin Liu, Giovanni Moreno, Haipan Shi
Huzhou Normal University, China
shihaipan226@163.com

We study the equivalence classes of slice–regular functions $f : \Omega \to \mathbb{H}$ on a symmetric slice domain Ω, and of their subclass made of polynomial slice–regular functions, with respect to the natural action of $\mathrm{PGL}(2, \mathbb{H})$ and its subgroups, by employing the twistor construction. In particular, we characterize slice–regular functions whose twistor lift is planar and belongs to a given orbit, and we find normal classes of slice–regular polynomials with respect to the action of a parabolic subgroup of $\mathrm{GL}(2, \mathbb{H})$.

Generalized Mandelbrot and Julia sets in arbitrary dimensions via a modified Clifford product

Neşet Deniz Turgay, Joan Lasenby, Cihan Güder, Arran Fernandez
Eastern Mediterranean University, Turkey
neset.turgay@emu.edu.tr

Fractals are mathematical objects characterized by non-integer, or fractal, dimensions. They arise naturally in many contexts, and approximations to fractals can often be observed in nature. Beyond their theoretical significance, fractals are also known for producing some of the most visually striking structures in mathematics. Classical examples include the Sierpiński triangle and the Menger sponge, whose constructions are relatively accessible, as well as the Mandelbrot set, where a simple iterative polynomial gives rise to highly intricate and complex behaviour.
The problem of extending the Mandelbrot set to higher dimensions has been explored in various forms, often with differing levels of mathematical rigour. We propose that Clifford algebras provide a natural and robust framework for such generalizations, as they extend the algebraic structure of the complex numbers $\mathbb{C}$ to higher dimensions.
Building on the work of Wareham and Lasenby (2011), we study the algebraic and topological properties of fractal sets defined in an arbitrary Clifford algebra $\mathrm{Cl}_{p,q}$ using a modified product operator $\odot$, given by

$$a \odot b = a e_1 b,$$

for all $a, b \in \mathrm{Cl}_{p,q}$. Although the vector space $\mathbb{R}^n$ is not a subalgebra under this product, it remains closed under the squaring operation. This allows us to define analogues of Mandelbrot and Julia sets via the iteration of the map

$$f_c(z) := z \odot z + c,$$

where $c, z \in \mathbb{R}^n \subset \mathrm{Cl}_{p,q}$.
We investigate structural properties of these generalized fractal sets and establish a connection with the Catalan numbers. Moreover, when $p = n$ and $q = 0$ we show that higher dimensional Mandelbrot set is compact and connected.

Parabolic Dirac operators and their fundamental solutions in parametric Clifford algebras

Eusebio Ariza, Judith Vanegas, Franklin Vargas
Yachay University, Ecuador
cvanegas@yachaytech.edu.ec

In this work, we presented a factorization for the non-stationary diffusion operator for anisotropic media $\tilde{\mathcal{L}} = \partial_t - div(B\nabla)$, where $B \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix. The factorization is given by $\tilde{\mathcal{L}} = \mathcal{D}^2_{x,t}$ where $\mathcal{D}_{x,t}$ is a parabolic Dirac operator defined on a parametric Clifford algebra.For the anisotropic heat diffusion operator and its factor, the parabolic Dirac operator, we find fundamental solutions, and for the last operator, a Cauchy-Pompeiu type integral representation is found. Integral operators derived from this integral representation allow us to pose mixed problems for the $\mathcal{D}_{x,t}$ operator.

References

[1] S. Bernstein, *Factorization of the nonlinear Schrödinger equation and applications*, Complex Variables and Elliptic Equations, volume 51, no. 5-6, pp. 429-452, 2006.

[2] P. Cerejeiras, U. Kähler, F. Sommen, *Parabolic Dirac operators and the Navier–Stokes equations over time-varying domains*, Mathematical methods in the applied sciences, volume 28, no.14, pages 1715-1724, 2005.

[3] N. Vieira, *Theory of the parabolic Dirac operators and its applications to non-linear differential equations*, Universidade de Aveiro, Portugal, 2005.

[4] W. Tutschke, C. J. Vanegas, *Clifford algebras depending on parameters and their applications to partial differential equations.* In: Some topics on value distribution and differentiability in complex and p-adic analysis, pages 430-450, 2008.

[5] J. Vanegas, F. Vargas, *On weighted Dirac operators and their fundamental solutions for anisotropic media*, Advances in Applied Clifford Algebras, volume 28, pp. 1-13, 2018.

[6] J. Vanegas, F. Vargas, *On weighted Dirac operators and their fundamental solutions*, Quaestiones Mathematicae, volume 43, no.3, pp. 383-393, 2020.

The Hartogs-Bochner extension for monogenic functions of several vector variables and the Dirac complex

Yun Shi, Wei Wang
Zhejiang University, China
wwang@zju.edu.cn

I will discuss the Dirac complex resolving several Dirac operators, which plays the fundamental role to investigate monogenic functions of several vector variables. The spaces in the Dirac complex are complicated irreducible modules of $\mathrm{GL}(k)\times\mathrm{Spin}(n)$. However, a simple characterization of the first four spaces is given and used to write down first three operators in the Dirac complex explicitly and to show this part to be an elliptic complex. Then the PDE method can be applied to obtain solutions to the non-homogeneous several Dirac equations under the compatibility condition, which implies the Hartogs' phenomenon for monogenic functions. Moreover, we find the boundary version of several Dirac operators and introduce tangentially monogenic functions, corresponding to tangential Cauchy-Riemann operator and CR functions in several complex variables, respectively, and establish the Hartogs-Bochner extension for tangentially monogenic functions on the boundary of a domain. If time permits, I will also discuss the Hessian complex of two octonionic variables, which is the octonionic version of $\partial\bar{\partial}$-complex.

Intrinsic q-radial vector derivatives and localized Fischer decompositions on radial algebras

Diana Schneiderová, Juan Bory-Reyes, Baruch Schneider, Yifan Zhang
University of Ostrava, Czech Republic
yifan.zhang@osu.cz

Radial algebras, introduced by Sommen as an algebra of abstract vector variables [2] and later developed as an abstract framework for orthogonal and Hermitian Clifford analysis [3], provide a dimension-free setting for the vector derivative and Fischer decomposition. In this talk, based on the preprint [1], we construct an intrinsic q-deformation of the vector derivative on radial algebras. In contrast with the coordinatewise approach obtained by replacing ordinary partial derivatives with one-variable Jackson derivatives, the present construction is formulated directly inside the radial algebra and therefore preserves the radial subalgebras generated by vector variables and their scalar products. For a distinguished vector variable x and a finite set of auxiliary variables $Y \subset S \setminus \{x\}$, we define a q-Cartan derivative $\partial_{x,q}^{Y}$ on $R(\{x\} \cup Y)$ in terms of the x-relative scalar variables

x^2 and $\{x, y\}$, $y \in Y$.
We prove two Fischer-type decomposition theorems in this setting. First, for the exterior Fischer operator, the corresponding anticommutator has a triangular structure with explicit resonance factors; after inverting these factors, one obtains a global Green operator and an exterior direct-sum decomposition. Second, using full left multiplication by x, we establish a monogenic Fischer decomposition after localization by finite-block determinants. We also describe the first denominator factors explicitly: the one-vector and two-vector factors are given in closed form, while the general determinant factors split according to x-support. Finally, a support-rank obstruction in degree zero shows that a universal unlocalized theorem for all real $0 < q < 1$ cannot hold without excluding q-resonances.
This work is co-funded by GAČR grant 25-16847S, and by University of Ostrava grant SGS05/PřF/2026.

References

[1] D. Barseghyan (Schneiderová), J. Bory-Reyes, B. Schneider, Y. Zhang, *Intrinsic q-radial vector derivatives and localized Fischer decompositions on radial algebras*, arXiv:2605.00775, 2026.

[2] F. Sommen, *An algebra of abstract vector variables*, Portugaliae Mathematica, 54(3), 287-310, 1997.

[3] H. De Schepper, A. Guzmán Adán, F. Sommen, *The radial algebra as an abstract framework for orthogonal and Hermitian Clifford analysis*, Complex Analysis and Operator Theory, 11, 1139-1172, 2017.

q-analytic functions and q-Fueter variables

<u>Martha Zimmermann</u>, Amedeo Altavilla, Swanhild Bernstein
TU Bergakademie Freiberg, Germany
martha-lina.zimmermann@math.tu-freiberg.de

We consider a q-deformation of the Fock space of holomorphic functions on $\mathbb{C}$ based on the definition of q-analyticity. This definition is based on the q-Cauchy-Riemann operator $D_{\bar{z}} = \frac{1}{2}\left(D_x^q + \mathrm{i}D_y^{1/q}\right)$, where $D_x^q f(x,y) = \frac{f(qx,y)-f(x,y)}{(q-1)x}$ is the q-deformed partial derivative with respect to x and $D_y^{1/q}$ is the q-deformed partial derivative with respect to y and deformation parameter $\frac{1}{q}$. In this context, we can define q-analytic monomials $z_q^n = (x + \mathrm{i}y)(x + \mathrm{i}qy)\dots(x + \mathrm{i}q^{n-1}y)$ and construct the associated q-Fock space as a Hilbert space with orthonomal basis $\{z_q^n/\sqrt{[n]_q!}\}_{n\geq 0}$.

To generalize this to the setting of Clifford analysis, we now introduce a new modified q-Dirac operator

$$D_q = e_0 \partial_0^{1/q} + \sum_{i=1}^{n} \partial_i^q e_i.$$

We can find q-monogenic functions and especially a q-analogue of the Fueter variables, structured similarly to the q-analytic monomials. The q-Fueter variables lead to a q-Cauchy-Kovalevskaya extension theorem.
Finally, we will also consider whether it is possible to find a q-deformed version of the Fueter-Sce theorem.

References

[1] A. Altavilla, S. Bernstein, M. Zimmermann, *Peculiar Behaviors of Functions in q-Calculus.* In Proceedings of ICHAA 2024: Hypercomplex Analysis and Its Applications, Trends in Mathematics, vol 9. Birkhäuser, Cham, 2025.

[2] A. Altavilla, S. Bernstein, M. Zimmermann, *q-Fock space of q-Analytic Functions and its Realization in* $L^2(\mathbb{C}; e^{-|z|^2}\,dx\,dy)$, arXiv.2511.09336, 2025.

[3] O. K. Pashaev, S. Nalci, *q-analytic functions, fractals and generalized analytic functions*, Journal of Physics A: Mathematical and Theoretical, Vol. 47, No. 4, 2014.

[4] H. Malonek, *A new hypercomplex structure of the Euclidean space* $\mathbb{R}^{m+1}$ *and the concept of hypercomplex differentiability*, Complex Variables, Theory and Application: An International Journal, pp. 25-33, Vol. 14, No. 1-4, 1990.

[5] A. De Martino, K. Diki, A. Guzmán Adán, *On the Connection between the Fueter-Sce-Qian Theorem and the Generalized CK-Extension*, Results in Mathematics, Vol. 78, No. 55, 2023.

Geometric algebra for modern signal processing: Foundations and frontiers

This session highlights cutting-edge research where Geometric Algebra (GA), including quaternions as well as related algebras like octonions and Okubo algebra, etc., provide a unifying mathematical framework for advanced signal processing. We will explore foundational advances that transcend traditional methods, focusing on novel GA-based integral transforms (including a.o. Fourier transforms, linear canonical transforms, wavelets, etc.), hypercomplex tensor decompositions, and generalized convolution theorems. A key theme is the application of these theoretical tools to solve complex, multidimensional problems in areas such as vector-sensor processing, image analysis, and geometric deep learning. The session aims to bridge theoretical mathematics and practical implementation, demonstrating how GA leads to more efficient, coherent, and intuitively derived solutions. We invite contributions on both the theoretical development of these tools and their application to demonstrate the potent synergy between abstract algebra and practical computation.

Session organisers: Anna Kit Ian Kou (University of Macau, China), Eckhard Hitzer (International Christian University, Japan).

Representation of faulted signals in three phase power system using conformal geometric algebra

Johanka Brdečková
Brno University of Technology, Czech Republic
209345@vutbr.cz

In power system engineering, voltage signals from a three-phase power system are analyzed for fault detection and monitoring. The values of voltage in each phase can be interpreted as coordinates of a point in $\mathbb{R}^3$. It allows us to see the signal as a spatial curve. In ideal case these points form a circle. Signal with a fault is a parametrization of an ellipse in $\mathbb{R}^3$. We use the concept of osculating circles to determine the parameters of such an ellipse and characterize the fault evolution through conformal transformations.
Having a set of measuring we would like to find the parameters of the corresponding ellipse. We may take triplets of following or near points and by outer product in Conformal Geometric Algebra find the approximate osculating circles. Among these, the smallest circle is identified as the hyper-osculating circle, that is a circle at the main vertex where the curvature reaches its maximum. This circle contains all the necessary information needed to characterize the ellipse.
We describe the evolution of the signal in time as a conformal transformation of the associated hyper-osculating circle.

References

[1] A.H. Eid, F.G. Montoya, *A Systematic and Comprehensive Geometric Framework for Multiphase Power Systems Analysis and Computing in Time Domain*, IEEE Access, 10, (132725-132741), 2022.

[2] J. Brdečková, J. Hrdina, *Notes on the losses compensation of a three-phase power system with dc offset based on geometric algebras*, Nonlinear Analysis: Modelling and Control, 30(2), 2025.

[3] L. Dorst, *The Construction of 3D Conformal Motions*, Math.Comput.Sci. 10, 97–113, 2016.

[4] J. Lasenby, H. Hadfield, A. Lasenby, *Calculating the Rotor Between Conformal Objects*, Adv. Appl. Clifford Algebras 29, 102, 2019.

[5] J. Ventura, J. Hrdina, A. Eid, M. Stodola, F.M. Arrabal-Campos, A. Alcayde, F.G. Montoya, *On the Fitting of Three-Phase Voltage Curves with Ellipses Using Geometric Algebra for Conics*, Proceedings - ICREPQ, 2024.

Poncelet's theorem with projective algebra

Oliver Conradt
Goetheanum, Switzerland
mas@goetheanum.ch

The 2^n-dimensional projective algebra Λ_n is a tool to describe metric-free concepts from projective geometry. [1] We will show how to proof Poncelet's Theorem in the non-metric context of projective geometry by applying projective algebra in the planar field Λ_3 and projective algebra in space Λ_4.

References

[1] O. Conradt, *Projective Geometry with Projective Algebra. Transition to Clifford Double Algebras and to Metric Geometries. Space and Counterspace*, Manuscript, submitted for publication, June 2026.

Algebraic vector structures of arbitrary finite dimensions

Wolf-Dieter Richter
University of Rostock, Germany
wolf-dieter.richter@uni-rostock.de

Adapting and modifying the approaches in [1] and [2], vector signals which evolve over time like generalized trigonometric functions, can be considered at any fixed time in two dimensions as elements of a generalized circle, and in higher dimensions as elements of a suitably defined sphere.
For studying such vectors with exclusively real components, we introduce vector-valued vector products and exponential functions in dimensions 2,3,4 and higher and base the definition of corresponding algebraic vector structures on these [3].
Making use of the new notion of vector division, we open up a field of geometric differential calculation which allows complementary considerations to [4].
Moreover, our approach leads us into a new world of generalized Euler formulas [5] and contributes some new aspects to vector-valued Fourier transformation that complements those in [6].
Throughout the talk, we place particular emphasis on the geometric properties and interpretation of all considered algebraic vector structures.

References

[1] D. Gabor, *Theory of communication*, J. Inst. Elect. Eng., vol. 93, pp. 429-457, 1946.

[2] B. Witten, S. Shragge, *Quaternion-based Signal Processing*, SEG/New Orleans, Annual Meeting, 2006.

[3] W.-D. Richter, *On complex numbers in higher dimensions*, Axioms, 11, 22, 2022.

[4] K.I. Kou, Y.-H. Xia, *Linear quaternion differential equations: basic theory and fundamental results*, arXiv.1510.02224v5, 2017.

[5] W.-D. Richter, *Vector representations of Euler's formula and Riemann's Zeta function*, Symmetry, 17, 1597, 2025.

[6] E. Hitzer, *The quaternion domain Fourier transform and its properties*, Adv.Appl.Clifford Algebras 26, 969-984, 2016.

Harmonic analysis

Harmonic Analysis has always played a key role in developing new methods and topics in both complex and hypercomplex analysis as well as in the broader application of Clifford algebras. This session is dedicated to recent advances in harmonic analysis in both two and higher-dimensional settings, with a particular focus on the interplay of algebraic, analytic and geometric methods. Topics will include, but are not limited to, Fourier transforms and related integral transforms over topological groups, symbolic calculi, applications to spectral theory, and functional and integral inequalities. Special emphasize will also be given to time-frequency analysis and its applications in signal and image processing. Contributions on the application of methods from harmonic analysis in machine and deep learning are also welcome.

Session organisers: Uwe Kähler (Universidade de Aveiro, Portugal), Jens Wirth (Universität Stuttgart, Germany).

Logarithmic uncertainty principle for Clifford wavelet transform

Hicham Banouh, **Anouar Ben Mabrouk**
University of Bouira, Algeria
h.banouh@univ-bouira.dz

This paper establishes a logarithmic uncertainty principle for the continuous Clifford wavelet transform within the non-commutative Clifford algebra framework. Merging uncertainty principles, wavelet theory, and Clifford analysis, we extend Beckner's classical logarithmic inequality [4] to Clifford-valued functions. For admissible Clifford mother wavelets and Schwartz class functions, we prove that :

$$\int_{Spin(n)}\int_{\mathbb{R}^+}\int_{\mathbb{R}^n} \ln|b|\,|T_\psi[f](a,b,s)|^2 dV(b)\frac{da}{a^{n+1}}ds + \frac{A_\psi}{(2\pi)^n}\int_{\mathbb{R}^n}\ln|\xi|\,|\hat{f}(\xi)|^2 dV(\xi) \geq \left(\frac{\Gamma'(\frac{n}{4})}{\Gamma(\frac{n}{4})}+\ln 2\right)A_\psi\|f\|_2^2,$$

where A_ψ is the wavelet admissibility constant. The proof combines Beckner's principle with integration over the wavelet parameters and utilizes key properties of the Clifford–Fourier transform [1,2]. This result refines previous Heisenberg-type inequalities [1,2] and contributes to harmonic analysis in Clifford algebras, with potential applications in signal and image processing where Clifford-valued representations arise.

References

[1] H. Banouh, A. Ben Mabrouk, M. Kesri, *Clifford Wavelet Transform and the Uncertainty Principle*, Adv. Appl. Clifford Algebras, 29, 106, 2019.

[2] H. Banouh, A. Ben Mabrouk, *A sharp Clifford wavelet Heisenberg-type uncertainty principle*, J. Math. Phys., 61, 093502, 2020.

[3] H. Banouh, *Uncertainty principle associated with the continuous Clifford wavelet Transform*, 12th International Conference on Clifford Algebras and Their Applications in Mathematical Physics (ICCA12), Hefei, China, 2020.

[4] W. Beckner, *Pitt's inequality and the uncertainty principle*, Proc. Amer. Math. Soc., 123(6), 1897–1905, 1995.

[5] G.B. Folland, A. Sitaram, *The uncertainty principle: A mathematical survey*, J. Fourier Anal. Appl., 3(3), 207–238, 1997.

Harmonic analysis over generalised q-Fock spaces

Paula Cerejeiras
University of Aveiro, Portugal
pceres@ua.pt

Quantum calculus was originally developed as a q-analog of classical differentiation. However, it soon became visible that it had applications in a large range of fields, such as quantum mechanics, number theory, the theory of special functions, and machine learning, among others.

More recently, quantum calculus has been shown to provide a scale of reproducing kernel Hilbert spaces (RKHS) that interpolate between two fundamental spaces, the Hardy space and the Fock space. This further allows for an interplay between the backward shift operator and the q-Jackson operator. In this talk, we explore this connection, the arising structural identities, and discuss some of its consequences.

Zero modes and Dirac-(logarithmic) Sobolev-type inequalities

Marianna Chatzakou, Uwe Kähler, Michael Ruzhansky
Ghent University, Belgium
fsttq@um.edu.mo

We study the decay rate of the zero modes of the Dirac operator with a matrix-valued potential that is considered here without any regularity assumptions, compared to the existing literature. For the Dirac operator and for Clifford-valued functions we prove the L^p-L^2 Dirac Sobolev inequality with explicit constant, as well as the L^p-L^q Dirac-Sobolev inequalities. We prove its logarithmic counterpart for $q = 2$, extending it to its Gaussian version of Gross, as well as show Nash and Poincaré inequalities in this setting, with explicit values for constants.

References

[1] M. Chatzakou, U. Kähler, M. Ruzhansky, *Zero modes and Dirac-(logarithmic) Sobolev-type inequalities*, Letters in Mathematical Physics, 116, Article 73, 2026.

Weyl operators and complex Itô-Hermite polynomials: From the classical to the q-setting

Kamal Diki
Ghent University, Belgium
kamal.diki@ugent.be

In this talk, we introduce the Weyl operator on the q-Fock space, also called the Arik–Coon space, and show that the Weyl operator we define satisfies the canonical commutation relations (CCR). To this end, we further investigate properties of the Fock space in the framework of q-calculus. First, we derive a new formula for the q-complex Itô–Hermite polynomials using properties of the q-extension of the backward shift operator. We then introduce and study q-analogs of the Segal–Bargmann transform and the Weyl operator. This talk is based on joint work with D. Alpay, P. Cerejeiras, A. De Martino, and U. Kaehler.

Very weak solutions for quaternionic Dirac operators with Hermitian distributional coefficients

Narciso Gomes
University of Cape Verde, Cape Verde
gomes.ng@gmail.com

We study a Cauchy problem involving quaternionic Dirac operators with time-dependent Hermitian distributional coefficients. Using a regularization procedure and suitable energy estimates, we introduce a framework for very weak solutions. We establish the existence and uniqueness of very weak solutions, as well as consistency with the classical theory under appropriate regularity assumptions.

Octonion phase retrieval: Intrinsic symmetry, sign-orbit geometry, and native Kaczmarz reconstruction

Ren Hu
Great Bay University, China
renhu@gbu.edu.cn

In this work, we study *octonion phase retrieval*: the recovery of a signal $\mathbf{x} \in \mathbb{O}^d$ from intensity measurements $y_\ell = \left|\langle \mathbf{a}_\ell, \mathbf{x}\rangle\right|^2$, $\ell = 1, \ldots, m$, where the products are taken in a fixed order dictated by octonion non-associativity.
We show that the commonly assumed global right unit-octonion ambiguity is not a universal symmetry of the measurement map: in the full ambient model, the only universal unit right multipliers are the central signs $\{\pm 1\}$, which singles out the sign-orbit distance as the intrinsic reconstruction metric. Building on this geometry, we develop a native octonion randomized Kaczmarz method (ORKM) without pseudo-real lifting — a non-associative counterpart of the complex phaseless Kaczmarz method — and establish its local linear convergence under a suitable regularity condition. We further relate the underlying octonionic Hilbert-module structure to Clifford modules. Experiments on synthetic data and multispectral images demonstrate faster convergence and higher reconstruction quality than Wirtinger-flow-type baselines. Taken together, these results frame octonion phase retrieval as a problem in harmonic analysis over a non-associative division algebra, in which the algebraic structure directly governs identifiability, intrinsic geometry, and reconstruction. (This is a joint work with Dr. P. Lian from Tianjin Normal University)

Poisson integral transforms associated with the Landau problem on the hyperbolic disk

Zouhair Mouayn
Université Sultan Moulay Slimane de Beni-Mellal, Morocco
mouayn@gmail.com

Two parameters family of Flensted-Jensen spherical functions $\phi_(\mu, \lambda)$ on the universal covering group of $U(1,1)$ are explicitly given and turn out to be, up to a phase factor, eigenfunctions of a B-weight Maass Laplacian operator on the Poincaré disk, associated with an eigenvalue parametrized by a complex number α. The correspondence : $(\mu, \lambda) - - > (B, \alpha)$ beteween parameters is explained. Conversely, for $\mathcal{R}\rceil(\alpha)$ different from 1/2, we characterize eigenfunctions of the operator under consideration, which are Poisson transforms of square integrable functions on the disk boundary. We also mention a technical difficulty while trying to obtain a similar characterization for the case $\mathcal{R}\rceil(\alpha) = 1/2$.

Fourier inequalities in Morrey and Campanato spaces

A. Debernardi Pinos, E. Nursultanov, S. Tikhonov
Autonomous University of Barcelona, Spain
adebernardipinos@gmail.com

Weighted norm inequalities in Morrey spaces have been widely studied for various classes of integral operators [1] (maximal operators, Riesz potentials, singular integrals, etc.) However, for integral transforms with oscillating kernels, and in particular for the Fourier transform, such inequalities have not been studied, in part due to the lack of the necessary tools.

In this talk we discuss recently obtained weighted norm inequalities for the Fourier transform in the context of Morrey and Campanato spaces [2]. More precisely, we focus on inequalities of the form

$$\|\widehat{f}\|_{X^{\lambda}_{p,q}} \lesssim \|f\|_{Y}, \tag{13}$$

where X is either a Morrey or Campanato space [3] and Y is an appropriate function space.

In the case of the Morrey space we discuss how to sharpen known Morrey-to-Lorentz estimates (that is, when Y is a Lorentz space; in fact, the approach is largely based on refined estimates involving Lorentz-type spaces). We also show that (13) cannot hold when both X and Y are Morrey spaces.

When X is a Campanato space, we prove that (13) holds for Y being the truncated Lebesgue space.

References

[1] V. I. Burenkov, *Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces. I*, Eurasian Math. J., 3(3), 11-32, 2012.

[2] A.D. Pinos, E. Nursultanov, S. Tikhonov, *Fourier inequalities in Morrey and Campanato spaces*, J. Funct. Anal., 287(7), Article No. 110522, 39p, 2024.

[3] Y. Sawano, G. Di Fazio, D. I. Hakim, *Morrey Spaces. Introduction and Applications to Integral Operators and PDE's. Volume I*, Monogr. Res. Notes Math., CRC Press, Boca Raton, FL, 2020.

The positivity of the Wigner distribution on LCA groups

Federico Riccardi
Polytechnic University of Turin, Italy
federico.riccardi@polito.it

The classical Hudson theorem states that Gaussian functions are the only functions on Euclidean space with everywhere nonnegative Wigner distribution. In this talk, we present a counterpart of Hudson's theorem for certain locally compact abelian groups. We show that, depending on the underlying group, functions with nonnegative Wigner distribution either do not exist or coincide —up to translations and scalar multiplication— with the so-called subcharacters of second degree. If time permits, we will also illustrate how a particular case of our result can be interpreted in terms of infinite quantum spin systems. This talk is based on joint work with Fabio Nicola.

References

[1] F. Nicola, F. Riccardi, *The Hudson theorem in LCA groups and infinite quantum spin systems*, arXiv:2507.13154, 2025.

Boundary value problems for Dirac operators on graphs

Alberto Richtsfeld
Stockholm University, Sweden
alberto.richtsfeld@math.su.se

We study the Dirac operator on metric graphs within the Bär–Ballmann framework for boundary value problems of first-order differential operators. This approach yields an index formula for general first-order operators on metric graphs. Focusing on self-adjoint boundary conditions for the Dirac operator on a complex line bundle, we derive an explicit spectral description in the case of equilateral graphs. We examine two distinguished classes of boundary conditions. The first is induced by permutation matrices and corresponds to decompositions of the graph into directed trails; the resulting spectrum determines key features of the decomposition, including the trail lengths. The second is defined via a directed incidence matrix. Here the spectrum is controlled by polynomials encoding information about the cycle structure of the directed graph, allowing, for instance, the direct computation of its girth.

Calculus on Spin groups and beyond

Jens Wirth
University of Stuttgart, Germany
jens.wirth@mathematik.uni-stuttgart.de

Underlying symmetries allow to construct adapted pseudodifferential and Fourier operator calculi for operators. This has been intensively studied over the past 15 years starting with the book [4] of M. Ruzhansky and V. Turunen and with many subsequent contributions.
The talk will present some of the key results for compact symmetries in particular for rotation and spin groups. Examples provided in the talk are based on [1], [2] and [3].

References

[1] P. Cerejeiras, U. Kähler, M. Ferreira, J. Wirth, *Global operator calculus on spin groups.* J. Fourier Anal. Appl., 29, no. 3, 2023.

[2] M. V. Ruzhansky, V. Turunen, J. Wirth, *Hörmander class of pseudo-differential operators on compact Lie groups and global hypoellipticity.* J. Fourier Anal. Appl., 20, no. 3, 476-499, 2014.

[3] M. V. Ruzhansky, J. Wirth, *Global functional calculus for operators on compact Lie groups*, J. Funct. Anal., 267, no. 1, 144-172, 2014.

[4] M. Ruzhansky, V. Turunen. *Pseudodifferential Operators and Symmetries*. Pseudo Diff. Oper., Vol. 2, Birkhäuser, Basel, 2010