Microeconomics I: Game Theory

Lecture 12:
Extensive Games with Perfect Information
(see Osborne, 2009, Sections 5.1, 6.1)

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Static games

Our previous study of situations of interaction has been restricted to cases in which the decision makers move simultaneously, i.e., in which the decision makers are not informed about the moves of the other decision makers when they are deciding about their moves.

Such situations are called static or simultaneous-move games.
Dynamic games

In the following lectures, we aim at overcoming this restriction.

We intend to model situations of interaction in which the decision makers move sequentially and are informed about previous moves of other decision makers.

Such situations are called dynamic of sequential-move games.
Dynamic games

In reality, many situations of interaction are of dynamic structure. Think, for example, of

- parlor games like TICTACTOE or board games like chess,
- wage bargaining processes between employers’ association and trade union,
- R&D investment races between companies,
- international economic relations.
Extensive game form

In order to study dynamic interactive decision-making we need game-theoretic concepts which capture its sequential structure.

The models representing situations of dynamic interaction are called extensive games.
Perfect and imperfect information

Models which study situations in which the interacting decision makers move sequentially and always alone, and they know the previous moves of the other decision-makers are referred to as extensive games with perfect information.

Models which study situations in which the interacting decision makers move sequentially, but do not know all previous or current moves of the other decision-makers are referred to as extensive game with imperfect information.
Perfect and imperfect information

Board games like chess or Nine Man Morris are prominent examples of dynamic games with perfect information.

Board games like Battleship or Scotland Yard are prominent examples of dynamic games with imperfect information.
Our plan of work

We start our analysis of dynamic games under the assumption that the decision-makers have perfect information. That is, they move alternately and are informed about the previous of the other decision-makers.

Afterwards we turn to games with imperfect information, in which decision-makers do not know always the previous and current moves of the other decision-makers.
Our plan of work

As usual, our first step is to set up a class of games which summarizes the rules of dynamic games. Then, in order to solve these games, we present and discuss prominent solution concepts.

But, before plunging into theory, let us play a game with sequential moves.
Ultimatum game

You are randomly selected to be either a proposer or a responder.

In case you are chosen as a proposer you receive € 10 (in our experiment, one chocolate eggs represents €1). As a proposer you must decide how much of this amount should be passed on to the responder. Only integer amounts between 0 and 10 are allowed to be offered to the responder.

In case you are chosen as a responder you are asked, for every conceivable offer of the proposer, whether you accept this offer or not. If you accept an offer, then this offer is implemented whenever it turns out to be the actual one. If you reject an offer, then neither proposer nor responder receives any payoff whenever this offer turns out to be the actual one.

After the group of proposers and responders have filled out the questionnaires, pairs of proposers and responders are randomly arranged.
Ultimatum game

This dynamic game with perfect information has been designed by Güth, Schmittberger and Schwarze (1982), who have called it ULTIMATUM GAME. It is one of the most popular economic experiments.

Werner Güth (February 2, 1944) is one of the best-known German economists who published over 250 articles in the fields of game theory and experimental economics. Currently, he is the acting director of the Max Planck Institute of Economics in Jena.
Further examples

Before introducing the formal framework for dynamic games with perfect information, let us consider another three prominent examples belonging to this class of games. It is the

- **Market Entry Game**, 
- **Centipede Game**, 
- **Cake Dividing Game**.
Market Entry game

An incumbent monopolist faces the possibility of entry of a challenger. The challenger must decide whether it should enter the market, or not. If it enters, the incumbent may either acquiesce or fight. The best outcome for the incumbent is that the challenger stays out of the market. In the case that the challenger enters the market, the incumbent prefers acquiescing to fighting. The best outcome for the challenger is the case in which it enters the market and the incumbent acquiesces. However, it prefers staying out of the market to the case in which it enters the market, but the incumbent fights.
Game tree of the Market Entry game

This dynamic game with perfect information may be visualized by the following game tree.

utility of challenger, utility of incumbent
Centipede game

Two players called $A$ and $B$ are involved in a process according to which they have alternately the opportunity to stop the game. Each player prefers the outcome when she stops the process in some period $t$ to that in which her opponent does so in period $t + 1$. However, if the game is stopped not before period $t + 2$ the player is always better off than in the case that she stops the game in period $t$. Let us suppose this process consists of four stages.
Game tree of the Centipede game

The four stage CENTIPEDE GAME may be visualized by following game tree.

utility of A, utility of B
Cake Dividing game

Twins $A$ and $B$ quarrel about their shares of a rectangular birthday sheet cake of length of one foot (ca. 30 centimeter). Both children like to eat as much cake as possible. To settle their quarrel the following procedure is proposed.

Child $A$ is invited to divide the cake into two pieces where $0 \leq x \leq 1$ measures in units of feet the distance from the left boundary of the cake to the dividing line chosen by $A$. After $A$ has split the cake, $B$ must decide whether he takes the left piece or the right piece of the cake.
Game tree of the Cake Dividing game

The CAKE DIVIDING GAME may be visualized by the following game tree.
Graphical description

Up to now, we have described graphically the setup of dynamic games with perfect information by so-called game trees.

Next, we introduce the formal representation of such games. The basic concept of the formal representation is that of a history. This concept captures the dynamics of these games.
Terminal history

Each **terminal history** \( h \) of a game is a finite sequence (i.e., \( h := (a^1, a^2, \ldots, a^k) \)) or an infinite sequence (i.e., \( h := (a^1, a^2, \ldots) \)) that represents a possible course of the game.

It describes a sequence of actions that may occur in the game. Its first component \( a^1 \) gives the action realized in period 1, its second component \( a^2 \) gives the action realized in period 2, and so on.
Finite and infinite terminal histories

A terminal history \( h := (a^t)^k_{t=1} \) is called

- **finite** if it is a finite sequence, i.e., if \( k < \infty \) holds,
- **infinite** if it is an infinite sequence, i.e., if \( k = \infty \) holds.

Obviously, a finite terminal history \( h := (a^t)^k_{t=1} \) ends after \( k \) periods whereas an infinite terminal history does not contain a terminal period.
Subhistories

A subhistory $h'$ of terminal history $h := (a^t)_{t=1}^k$ is either

- the empty sequence $\emptyset$, containing no element and representing the start of the game, or
- a sequence $(a^t)_{t=1}^l$ containing the first $1 \leq l \leq k$ components of the history $h$. 
Proper subhistory

Note, by definition, a terminal history $h$ is a subhistory of itself.

A subhistory $h'$ of terminal history $h$ is called a **proper subhistory** of $h$ whenever it is not equal to $h$. 
Histories of dynamic games

A sequence of actions that is a subhistory of some terminal history is called simply a \textit{history} of the game.

A sequence of actions that is a proper subhistory of some terminal history is called simply a \textit{non-terminal history} of the game.
Set of histories

Henceforth, we take capital letter

- $T$ to denote the set of terminal histories of the dynamic game.
- $H$ to denote the set of histories of the dynamic game.
- $Z$ to denote the set of non-terminal histories of the dynamic game.

Obviously, $H = T \cup Z$ holds.
Histories of Market Entry Game

**Exercise:** Consider the Market Entry Game. Determine, for this game, the following sets of histories!

- set of terminal histories \( T := \)
- set of histories \( H := \)
- set of non-terminal histories \( Z := \)
Histories of Centipede Game

**Exercise:** Consider the Centipede Game. Determine, for this game, the following sets of histories!

= set of terminal histories

= set of histories

= set of non-terminal histories
Histories of Cake Dividing Game

**EXERCISE:** Consider the Cake Dividing Game. Determine, for this game, the following sets of histories!

- set of terminal histories \( T := \)
- set of histories \( H := \)
- set of non-terminal histories \( Z := \)
Extensive game with perfect information

Based on the concept of a history, we are able to set up the class of games which models dynamic games with perfect information.

This class of games is known as extensive games with perfect information.
**Definition 12.1**

An extensive game with perfect information \( \Gamma := (I, T, P, (\succsim_i)_{i \in I}) \) consists of

- a non-empty finite set \( I \) of players
- a non-empty set \( T \) of terminal histories
- a player function \( P : Z \to I \) assigning a player to each non-terminal history
- for each player \( i \in I \), a preference relation \( \succsim_i \) on \( T \) which is representable by utility function \( U_i \)
Extensive games with perfect information

According to the previous definition an extensive game with perfect information consists of following four components:

- set of players,
- set of terminal histories that lists all sequences of action that might occur in the game,
- a player function that determines, for every non-terminal history, which player moves next,
- for every player, a preference relation on the set of terminal histories
Components of the Market Entry Game

EXERCISE: Describe the MARKET ENTRY GAME as an extensive game with perfect information.

set of players \( I := \)

set of terminal histories \( T := \)

player function \( P : \)

players’ preferences \( U_{Challenger} : \)

\( U_{Incumbent} : \)
Components of the Centipede Game

**EXERCISE:** Describe the **CENTIPEDE GAME** as an extensive game with perfect information

- **set of players** \( I := \)
- **set of terminal histories** \( T := \)
- **player function** \( P : \)
- **players’ preferences** \( U_A : \)
  \( U_B : \)
Components of the Cake Dividing Game

**EXERCISE:** Describe the *Cake Dividing Game* as an extensive game with perfect information

- set of players: \( I := \)
- set of terminal histories: \( T := \)
- player function: \( P : \)
- players’ preferences: \( U_A : \), \( U_B : \)
Length of a history

Let $\Gamma$ be an extensive game with perfect information.

The **length of history** $h$ of $\Gamma$ is the number of periods recorded in this history. Henceforth, it is denoted by $\ell(h)$.

We stipulate that $\ell(\emptyset) := 0$. Obviously, if $h := (a^t)_{t=1}^k$, then $\ell(h) = k$.

The set of all histories of length $k$ in $\Gamma$ is denoted by $H^k$. Obviously, $H^0 = \{\emptyset\}$ holds.
Length of a game

Let $\Gamma$ be an extensive game with perfect information.

The **length of game** $\Gamma$ is defined as the length of the longest terminal history in $\Gamma$. In the case that for every $k \in \mathbb{N}$ there is a terminal history whose length is larger than $k$, the length of infinity is attributed to game $\Gamma$.

Henceforth, the length of game $\Gamma$ is denoted by $\ell(\Gamma)$. 
Finite and infinite horizon

An extensive game with perfect information is said to have **finite horizon** if its length $\ell(\Gamma)$ is finite (i.e., there exists some $k \in \mathbb{N}$ so that $\ell(h) \leq k$ holds for every $h \in T$).

An extensive game with perfect information is said to have **infinite horizon** if its length is infinite (i.e., there is no $k \in \mathbb{N}$ so that $\ell(h) \leq k$ holds every $h \in T$).
Finite and infinite extensive games

An extensive game with perfect information is called **finite** if it has finite horizon and the set of terminal histories is finite.

An extensive game with perfect information which is not finite is called **infinite**.
Exercise: Length of games

Exercise: Determine the length of the Market Entry Game, Centipede Game, and Cake Dividing Game. Which of them have a finite horizon, which are finite?
Updated histories

Consider an extensive game $\Gamma := (I, T, P, (\succsim_i)_{i \in I}$ with perfect information.

Let $h' := (a'_t)_{t=1}^k$ be a history of $\Gamma$ and $h'' := (a''_t)_{t=1}^l$ be a sequence of actions. Then history $h := (h', h'') := (a_t)_{t=1}^{k+l} \in H$ specified by

$$a_t := \begin{cases} 
  a'_t & \text{if } t \leq k, \\
  a''_{t-s} & \text{if } t > k,
\end{cases}$$

for every $t \in \{1, \ldots, k + l\}$ is called the history in which history $h'$ is updated by action sequence $h''$. 
Updated histories

Let $h' := (a'_t)_{t=1}^k \in H$ be a history of game $\Gamma$ and element $a$ some action. Then history $h := (h, a) := (a_t)_{t=1}^{k+1} \in H$ specified by

$$a_t := \begin{cases} a'_t & \text{if } t \leq k, \\ a & \text{if } t = k + 1, \end{cases}$$

for every $t \in \{1, \ldots, k + 1\}$ is called the history in which history $h'$ is updated by action $a$. 
Available actions

Let \( h \in H^k \) be a non-terminal history of length \( k \). Then

\[
A(h) := \{ a : (h, a) \in H^{k+1} \}
\]

is the set of actions available for the player who moves after history \( h \).
Action set

Consider some player $i \in I$ of extensive game $\Gamma$ with perfect information.

The set of all non-terminal histories after which player $i$ moves is denoted by

$$Z_i := \{ h : P(h) = i \} .$$

The set of all actions that player $i$ may choose during the game is denoted by

$$A_i := \bigcup_{h \in Z_i} A(h)$$

It is called player $i$’s action set in $\Gamma$. 
**Exercise**: Consider the Market Entry Game. Determine the set of available actions after every non-terminal history of this game.

**Answer:**
Action sets of Centipede Game

**EXERCISE:** Consider the *Centipede Game*. Determine the set of available actions after every non-terminal history of this game.

**ANSWER:**
Action sets of Cake Dividing Game

Exercise: Consider the Cake Dividing Game. Determine the set of available actions after every non-terminal history of this game.

Answer: