Microeconomics I: Game Theory

Lecture 14:

Subgame Perfect Nash Equilibrium

(see Osborne, 2009, Sections 5.4 and 5.5)

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Subgame perfect Nash equilibrium

The solution concept of subgame perfect Nash equilibrium is a refinement of the solution concept of Nash equilibrium.

In order motivate this new solution concept, let us reconsider the Market Entry Game.
Reconsidering the Market Entry Game

An incumbent monopolist faces the possibility of entry of a challenger. The challenger must decide whether it should enter the market, or not. If it enters, the incumbent may either acquiesce or fight. The best outcome for the incumbent is that the challenger stays out of the market. In the case that the challenger enters the market, the incumbent prefers acquiescing to fighting. The best outcome for the challenger is the case in which it enters the market and the incumbent acquiesces. However, it prefers staying out of the market to the case in which it enters the market, but the incumbent fights.
Game tree of the Market Entry game

The Market Entry Game may be visualized by the following game tree.

Challenger

Incumbent

utility of challenger, utility of incumbent

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Nash equilibria of the Market Entry Game

To figure out the Nash equilibria of the Market Entry Game, the game is transformed in its strategic form.

Remember that the Nash equilibria of this form constitute the Nash equilibria of the Market Entry Game.
Strategic form of the Market Entry Game

The strategic form of the Market Entry Game is depicted by the below bimatrix.

<table>
<thead>
<tr>
<th>Challenger</th>
<th>Incumbent acquiesce if challenger chooses in</th>
<th>Incumbent fight if challenger chooses in</th>
</tr>
</thead>
<tbody>
<tr>
<td>out</td>
<td>1,2</td>
<td>1,2</td>
</tr>
<tr>
<td>in</td>
<td>2,1</td>
<td>0,0</td>
</tr>
</tbody>
</table>
Nash equilibria of the Market Entry Game

The Nash equilibria have been detected by the best response correspondences as depicted below.

<table>
<thead>
<tr>
<th>Challenger</th>
<th>Incumbent acquiesce if challenger chooses in</th>
<th>Incumbent fight if challenger chooses in</th>
</tr>
</thead>
<tbody>
<tr>
<td>out</td>
<td>1,2</td>
<td>1,2</td>
</tr>
<tr>
<td>in</td>
<td>2,1</td>
<td>0,0</td>
</tr>
</tbody>
</table>
Nash equilibria of the Market Entry Game

As can be easily seen from the previous bimatrix, the Market Entry Game has two Nash equilibria, namely strategy profile \((s^*_C, s^*_I)\) consisting of strategies

\[
s^*_C(\emptyset) := \text{out}, \quad s^*_I(\text{in}) := \text{fight},
\]

and the strategy profile \((s^{**}_C, s^{**}_I)\) consisting of strategies

\[
s^{**}_C(\emptyset) := \text{in}, \quad s^{**}_I(\text{in}) := \text{acquiesce}.
\]

In the following, we examine these strategy profiles and discuss whether they describe “reasonable” behavior.
Are all Nash equilibria reasonable?

Nash equilibrium \((s_C^*, s_I^*)\) prescribes that the potential challenger abstains from entering the market and the incumbent would fight if the potential challenger had chosen to enter the market. This Nash equilibrium is depicted in blue in below game tree.

However, does this Nash equilibrium describe reasonable decisions?
Are all Nash equilibria reasonable?

Suppose the challenger would - contrary to its initial plan of action - enter the market.

Does it make sense to the incumbent to respond with fighting?

Is its threat of fighting credible?

\[
\begin{array}{c|cc}
\text{Challenger} & \text{in} & \text{out} \\
\hline
\text{Incumbent} & \text{acquiesce} & 2,1 \\
& \text{fight} & 0,0 \\
\end{array}
\]
Are all Nash equilibria reasonable?

**Observation:** Nash equilibrium \((s_c^*, s_i^*)\) is suspect.

If incumbent observes that challenger has entered the market, incumbent should acquiesce. It would be worse off if instead it chooses to fight. Moreover, challenger should expect such behavior of incumbent in case it enters the market, and so challenger should do it.

The Nash equilibrium \((s_c^*, s_i^*)\) relies on an “empty threat” by incumbent to fight if challenger enters market. This threat is empty because incumbent would never wish to carry it out.
Subgame perfect Nash equilibrium

To exclude Nash equilibria resting on empty threats the additional requirement that the player’s strategies must be optimal in any part of the game is imposed.

Nash equilibria satisfying this requirement are termed subgame perfect Nash equilibria.
Subgame perfect Nash equilibrium

The solution concept of subgame perfect Nash equilibrium has been introduced by the German mathematician and economist Reinhard Selten (1965).

Reinhard Selten (born on October 5, 1930) received, together with John C. Harsanyi and John F. Nash, the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel in 1994 for their “pioneering analysis of equilibria in the theory of noncooperative games”.
Subgame perfect Nash equilibrium

As already mentioned, subgame perfect Nash equilibria are Nash equilibria which prove to be optimal for the players in every part of the game, i.e., after every possible history.

A part of an extensive game is referred to as a subgame. In the following, we introduce formally this notion.
Consider some extensive game \( \Gamma := (I, T, P, (\succeq_i)_{i \in I}) \) with perfect information, and let \( h \) be a non-terminal history of \( \Gamma \).

Verbally, the subgame \( \Gamma(h) \) following history \( h \) is the part of game \( \Gamma \) which contains only the branches of \( \Gamma \) that originate from branch \( h \).
Subgames of the Market Entry Game

The Market Entry Game has subgames $\Gamma(\emptyset)$ and $\Gamma(in)$.

$\Gamma(\emptyset)$

$\Gamma(in)$
Subgame

Consider some extensive game $\Gamma := (I, T, P, (\succeq_i)_{i \in I})$ with perfect information, and let $h$ be a non-terminal history of $\Gamma$.

The subgame $\Gamma(h)$ following history $h$ is like the entire game $\Gamma$ an extensive game with perfect information. Its components

$$\Gamma(h) := (I|_h, T|_h, P|_h, (\succeq_i|_h)_{i \in I})$$

are defined as follows.
Histories in subgames

Consider some extensive game $\Gamma := (I, T, P, (\succ_i)_{i \in I})$ with perfect information, and let $h$ be a non-terminal history of $\Gamma$.

Then,

$$Z|_h := \{ h' : (h, h') \in Z \}$$

is the set of sequences of actions which might follow history $h$ (including the empty sequence $\emptyset$), but do not terminate the game.

It is called the set of non-terminal histories in subgame $\Gamma(h)$.
Terminal histories in subgames

Consider some extensive game $\Gamma := (I, T, P, (\succeq_i)_{i \in I})$ with perfect information, and let $h$ be a non-terminal history of $\Gamma$.

Then,

$$T|_h := \{ h' : (h, h') \in T \}$$

is the set of sequences of actions which might follow history $h$ (including the empty sequence $\emptyset$) and which terminate the game.

It is called the set of terminal histories in subgame $\Gamma(h)$. 
Players in subgames

Consider some extensive game $\Gamma := (I, T, P, (\succ_i)_{i \in I})$ with perfect information, and let $h$ be a non-terminal history of $\Gamma$.

Then,

$$I|_h := \{P(h, h') : h' \in Z|_h\}$$

is the set of players who are required to move after some non-terminal update of history $h$.

It is called the **set of players in subgame** $\Gamma(h)$. 
Player function of subgames

Consider some extensive game $\Gamma := (I, T, P, (\succsim_i)_{i \in I})$ with perfect information, and let $h$ be a non-terminal history of $\Gamma$.

Then, mapping $P|_h$ determined by

$$P|_h(h') := P(h, h')$$

for every $h' \in Z|_h$ is the mapping that assigns to each non-terminal update of history $h$ the player whose turn it is.

It is called the player function of subgame $\Gamma(h)$. 
Preference relations in subgames

Consider some extensive game $\Gamma := (I, T, P, (\succeq_i)_{i \in I})$ with perfect information, and let $h$ be a non-terminal history of $\Gamma$.

Then, preference relation $\succeq_i|_h$ of player $i \in I|_h$ on $T|_h$ is representable by utility function $U_i|_h$ determined by

$$U_i|_h(h') := U_i(h, h')$$

for every $h' \in T|_h$ where $U_i$ is the utility function representing preference relation $\succeq_i$.

It is called player $i$’s preference relation in subgame $\Gamma(h)$. 
Subgame

Definition 14.1

Let $\Gamma := (I, T, P, (\succeq_i)_{i \in I})$ be an extensive game with perfect information and $h \in Z$ a non-terminal history of $\Gamma$.

The subgame $\Gamma(h)$ following history $h$ is the extensive game with perfect information whose components are

$$\Gamma(h) := (I|_h, T|_h, P|_h, (\succeq_i|_h)_{i \in I}).$$
Subgames of the Market Entry Game

Example: Market Entry Game consists of subgames

- \( \Gamma(\emptyset) = \Gamma \) (i.e., the entire game)
- \( \Gamma(in) = (I|_{in}, T|_{in}, P|_{in}, (\succsim_i|_{in})_{i \in I}) \), where
  - \( I|_{in} = \{ \text{Incumbent} \} \),
  - \( T|_{in} = \{(\text{acquiesce}), (\text{fight})\} \),
  - \( P|_{in} \) is the mapping assigning \( P|_{in}(\emptyset) = \text{Incumbent} \),
  - \( \succsim_{\text{Incumbent}}|_{in} \) is representable by utility function \( U_{\text{Incumbent}}|_{in} \) for which \( U_{\text{Incumbent}}|_{in}(\text{acquiesce}) = 1 \) and \( U_{\text{Incumbent}}|_{in}(\text{fight}) = 0 \) hold.
Strategies in subgames

To introduce the solution concept of subgame perfect Nash equilibrium it is also imperative to specify the players’ strategy sets in subgames. This is accomplished next.
Strategies in subgames

Consider some extensive game \( \Gamma := (I, T, P, (\succcurlyeq_i)_{i \in I}) \) with perfect information, and let \( h \) be a non-terminal history of \( \Gamma \).

Then, for every player \( i \in \mathcal{I}_h \),

\[
Z_i |_h := \{ h' \in Z | h : P |_h (h') = i \}
\]

is the set of all non-terminal histories in subgame \( \Gamma(h) \) in which it is \( i \)'s turn to move.
Strategies in subgames

Consider some extensive game $\Gamma := (I, T, P, (\succeq_i)_{i \in I})$ with perfect information, and let $h$ be a non-terminal history of $\Gamma$.

Then, for every player $i \in I | h$, mapping $s_i | h$ determined by

$$s_i | h(h') := s_i(h, h')$$

for every $h' \in Z_i | h$ is called the strategy induced by strategy $s_i$ on subgame $\Gamma(h)$. 
Strategies in subgames

The induced strategy $s_i|_h$ on subgame $\Gamma(h)$ gives, for every non-terminal history of subgame $\Gamma(h)$, the action player $i$ chooses according to her strategy $s_i$.

It is also termed the restriction of strategy $s_i$ on history $h$. 
Subgame perfect Nash equilibrium

Based on the previous specifications, the solution concept of subgame perfect Nash equilibrium may be presented.

Verbally, a subgame perfect Nash equilibrium is a strategy profile that induces a Nash equilibrium in every subgame of the extensive game.
Subgame perfect Nash equilibrium

Definition 14.2

Let $\Gamma := (I, T, P, (≿_i)_{i \in I})$ be an extensive game with perfect information.

A strategy profile $s^* := (s^*_i)_{i \in I}$ of $\Gamma$ is called a subgame perfect Nash equilibrium of $\Gamma$ if and only if, for every non-terminal history $h \in Z$, its restriction $s^*|_h := (s^*_i|_h)_{i \in I}$ is a Nash equilibrium of subgame $\Gamma(h)$. 
Subgame perfect Nash equilibrium

Since a subgame perfect Nash equilibrium generates a Nash equilibrium in every subgame, it generates, in particular, a Nash equilibrium in the entire game (which is the subgame following the empty history).

Hence, we obtain the following result.
Theorem 14.3

Let \( \Gamma := (I, T, P, (\succsim_i)_{i \in I}) \) be an extensive game with perfect information.

Every subgame perfect Nash equilibrium of \( \Gamma \) is also a Nash equilibrium of \( \Gamma \).
QUESTION: Which Nash equilibria of the Market Entry Game are subgame perfect?

To answer this question we have to check which of these strategy profiles generates a Nash equilibrium in every subgame of the Market Entry Game.
### Strategic form of subgames

**The strategic form of subgame $\Gamma(\emptyset)$ is**

<table>
<thead>
<tr>
<th>Challenger</th>
<th>out</th>
<th>in</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Incumbent</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>acquiesce if in occurs</td>
<td>1,2</td>
<td>2,1</td>
</tr>
<tr>
<td>fight if in occurs</td>
<td>1,2</td>
<td>0,0</td>
</tr>
</tbody>
</table>

**The strategic form of subgame $\Gamma(in)$ is**

<table>
<thead>
<tr>
<th>Challenger</th>
<th>acquiesce</th>
<th>fight</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Incumbent</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
Nash equilibria in subgames

- Nash equilibria of subgame $\Gamma(\emptyset)$ are depicted in blue

  \[
  \begin{array}{c|cc}
  \text{Challenger} & \text{out} & \text{in} \\
  \hline
  \text{acquiesce if in occurs} & 1,2 & 2,1 \\
  \text{fight if in occurs} & 1,2 & 0,0 \\
  \end{array}
  \]

- Nash equilibrium of subgame $\Gamma(\text{in})$ is depicted in blue

  \[
  \begin{array}{c|c}
  \text{Incumbent} & \text{acquiesce} & \text{fight} \\
  \hline
  1 & 0 \\
  \end{array}
  \]
Subgame perfectness in Market Entry Game

**QUESTION:** Is strategy profile $s^* := (s_c^*, s_i^*)$ a subgame perfect Nash equilibrium of the Market Entry Game?

**ANSWER:** Consider subgame $\Gamma(in)$. Incumbents’s strategy $s_i^*$ prescribes action $s_i^*(in) = \text{fight}$ for this subgame. However this action is not a Nash equilibrium of it.

Hence, strategy profile $s^*$ does not induce a Nash equilibrium for every subgame of the Market Entry Game, and thus it is not a subgame perfect Nash equilibrium.
Subgame perfectness in Market Entry Game

**Question:** Is strategy profile \( s^{**} := (s_C^{**}, s_I^{**}) \) a subgame perfect Nash equilibrium of the Market Entry Game?

**Answer:** Consider subgame \( \Gamma(in) \). Incumbent’s strategy \( s_I^{**} \) prescribes action \( s_I^{**}(in) = acquiesce \) for this subgame. This action is a Nash equilibrium of it.
SUBGAME PERFECTNESS IN MARKET ENTRY GAME

QUESTION: Is strategy profile \( s^{**} := (s^{**}_C, s^{**}_i) \) a subgame perfect Nash equilibrium of the MARKET ENTRY GAME?

ANSWER: (continued) Consider subgame \( \Gamma(\emptyset) \). Challenger’s strategy \( s^{**}_C \) prescribes to enter the market, and incumbent’s strategy \( s^{**}_i \) is the rule to acquiesce if the challenger enters the market. Obviously, this strategy profile is a Nash equilibrium of subgame \( \Gamma(\emptyset) \).
Subgame perfectness in Market Entry Game

**Question**: Is strategy profile $s^{**} := (s^{**}_C, s^{**}_I)$ a subgame perfect Nash equilibrium of the *Market Entry Game*?

**Answer**: (continued) Yes! As just shown, strategy profile $s^{**} := (s^{**}_C, s^{**}_I)$ induces a Nash equilibrium in every subgame of the *Market Entry Game*.
Finding subgame perfect Nash equilibrium

To determine the subgame perfect Nash equilibria of the Centipede Game and of the Cake Dividing Game, one can proceed as we did in the Market Entry Game.

That is to say, to consider the strategic form of each subgame and examine whether the strategy profile induces a Nash equilibrium for each of those subgames.
Backward induction procedure

However, in extensive games with perfect information and finite horizon, subgame perfect Nash equilibria may be found more directly by a procedure known as the backward induction procedure.
Backward induction procedure

Consider an extensive game $\Gamma$ with perfect information and finite horizon $k := \ell(\Gamma) < \infty$. The backward induction procedure on $\Gamma$ works as follows.

Step 1: Consider the subgames of length 1. Find all optimal actions of the players who move in these subgames. Pick, for each of those subgames, one of the optimal actions.

Step 2: Consider the subgames of length 2. Given the optimal actions picked in step 1, find all optimal actions of the players who move first in these subgames. Pick, for each of those subgames, one of the optimal actions.

\[ \vdots \]

Step $n$: Consider the subgames of length $n \leq k$. Given the optimal actions picked in the previous steps, find the optimal actions of the players who move first in these subgames. Pick, for each of those subgames, one of the optimal actions. Continue in this way until $n = k$.

HINT: Repeat the steps 1 to $k$ for every possible combination of optimal actions.
Backward induction procedure

To sum up, the backward induction procedure specifies the strategy profiles $s^* := (s^*_i)_{i \in I}$ which have the following property:

For every non-terminal history $h$, action $s^*_{P(h)}(h)$ is one of the best actions that player $P(h)$, who moves first in subgame $\Gamma(h)$, may choose given optimal actions in subgames with smaller length.

Such strategy profiles are called simply strategy profiles specified by the procedure of backward induction.
Theorem 14.4

Let $\Gamma$ be an extensive game with perfect information and finite horizon. The set of subgame perfect Nash equilibria of $\Gamma$ is equal to the set of strategy profiles specified by the procedure of backward induction.
Existence of subgame perfect equilibrium

Note that, in finite extensive games with perfect information, the player who moves first in a subgame has only a finite number of possible actions at her disposal.

Thus, in such a game, the procedure of backward induction isolates, in every subgame, at least one optimal action for the first mover. Therefore, at least one strategy profile may be specified by the backward induction procedure.

Together with the previous theorem, this implies that the game has at least one subgame perfect Nash equilibrium.
Existence of subgame perfect equilibrium

Theorem 14.5

Every finite extensive game with perfect information has a subgame perfect Nash equilibrium.
EXERCISE: Apply the procedure of backward induction to find all subgame perfect Nash equilibria of the MARKET ENTRY GAME.
Backward induction in Market Entry Game

**Step 1** Find the best actions of the players who move in subgames of length 1.

Obviously, the Market Entry Game has only one subgame of length 1, namely subgame \( \Gamma(in) \) after history \( (in) \).

The best action of the player in this subgame (i.e., of the incumbent) is to acquiesce.

\[
\begin{array}{c|c|c}
\text{Challenger} & \text{Incumbent} & \text{Outcome} \\
\text{in} & \text{acquiesce} & 2,1 \\
\text{out} & \text{fight} & 0,0 \\
\end{array}
\]
Backward induction in Market Entry Game

Step 1 Find the best actions of the players who move in subgames of length 1.

In formal terms, the backward induction procedure implies that, in a subgame perfect Nash equilibrium \((s_C, s_I)\), the incumbent’s strategy \(s_I\) must satisfy

\[ s_I(\text{in}) = \text{acquiesce} . \]
Backward induction in Market Entry Game

Step 2  Taking the best actions in subgames of length 1 as given, find the best actions of the players who move first in subgames of length 2.

Obviously, entire game $\Gamma$ is the only subgame of the Market Entry Game of length 2.

Given that incumbent chooses to acquiesce, the best action of challenger is to enter the market.
Backward induction in Market Entry Game

Step 2 Taking the best actions in subgames of length 1 as given, find the best actions of the players who move first in subgames of length 2.

In formal terms, the procedure of backward induction also implies, that in subgame perfect Nash equilibrium \((s_C, s_I)\) the challenger’s strategy must satisfy

\[ s_C(\emptyset) = \text{in} . \]
Backward induction in Market Entry Game

Since the length of the Market Entry Game is equal to two, the initial decision node of the game is reached after step 2, and thus the strategies of the players are completely specified.

Summing up, the procedure of backward induction applied on the Market Entry Game has yielded strategy profile \((s_C, s_I)\), whose strategies are given by

\[
  s_C(\emptyset) = \text{in} \quad \text{and} \quad s_I(\text{in}) = \text{acquiesce}.
\]
Backward induction in Market Entry Game

The subgame perfect Nash equilibrium is visualized by the blue lines in the Market Entry Game tree.

- Challenger
  - Incumbent
    - acquiesce: 2,1
    - fight: 0,0
  - out: 1,2
Subgame perfectness in Market Entry Game

According to Theorem 14.4, the set of strategy profiles resulting from the backward induction procedure corresponds to the set of subgame perfect Nash equilibria.

Hence, only Nash equilibrium \((s^*_C, s^*_I)\) proves to be a subgame perfect Nash equilibrium of the Market Entry Game.
Subgame perfectness in Centipede Game

EXERCISE: Find all subgame perfect Nash equilibria of the CENTIPEDE GAME!

▷ To master this exercise we apply the backward induction procedure on the CENTIPEDE GAME.
Backward induction in Centipede Game

Step 1 Find the best actions of the players who move in the subgames of length 1.

Obviously, subgame $\Gamma(c, c, c)$ after history $(c, c, c) := (\text{continue, continue, continue})$ is the only subgame of length 1 of the Centipede Game.
Step 1  Find the best actions of the players who move in the subgames of length 1.

As can be easily seen, the best action of player \( B \) in subgame \( \Gamma(c, c, c) \) is to stop.
Backward induction in Centipede Game

Step 2 Taking the best actions in subgames of length 1 as given, find the best actions of the players who move first in subgames of length 2.

Obviously, subgame $\Gamma(c, c)$ after history $(c, c) := (\text{continue, continue})$ is the only subgame of length 2 of the Centipede Game.
Backward induction in Centipede Game

Step 2 Taking the best actions in subgames of length 1 as given, find the best actions of the players who move first in subgames of length 2.

Given that player $B$ stops after history $(c, c, c)$, the best action of player $A$ who moves first in subgame $\Gamma(c, c)$ is to stop, too.
Backward induction in Centipede Game

Step 3 Taking the best actions in subgames of length less than 3 as given, find the best actions of the players who move first in subgames of length 3.

Obviously, subgame $\Gamma(c)$ after history $(c) := (\text{continue})$ is the only subgame of length 3 of the Centipede Game.
Step 3 Taking the best actions in subgames of length less than 3 as given, find the best actions of the players who move first in subgames of length 3.

Given that player A stops after history \((c, c)\), the best action of player B who moves first in subgame \(\Gamma(c)\) is to stop, too.
Backward induction in Centipede Game

Step 4  Taking the best actions in subgames of length less than 4 as given, find the best actions of the players who move first in subgames of length 4.

Obviously, the entire game $\Gamma$ is the only subgame of length 4 of the Centipede Game.
Backward induction in Centipede Game

Step 4 Taking the best actions in subgames of length less than 4 as given, find the best actions of the players who move first in subgames of length 4.

Given that player B stops after history (c), the best action of player A who moves first in game Γ is to stop, too.

\[
\begin{align*}
\Gamma(\emptyset) \\
A &\quad \text{continue} & B &\quad \text{continue} & A &\quad \text{continue} & B &\quad \text{continue} & 5,3 \\
\bullet &\quad \text{stop} & \bullet &\quad \text{stop} & \bullet &\quad \text{stop} & \bullet &\quad \text{stop} \\
1,0 & & 0,2 & & 3,1 & & 2,4 &
\end{align*}
\]
Backward induction in Centipede Game

Since the length of the CENTIPEDE GAME is equal four the initial decision node of the game is reached after step 4, and thus the strategies of the players are completely specified.

Summing up, the procedure of backward induction applied on the CENTIPEDE GAME result in strategy profile \((s^*_A, s^*_B)\) which consists of strategies

\[
\begin{align*}
    s^*_A(\emptyset) & = \text{stop}, \\
    s^*_A(c, c) & = \text{stop}, \\
    s^*_B(c) & = \text{stop}, \\
    s^*_B(c, c, c) & = \text{stop}.
\end{align*}
\]
Backward induction in Centipede Game

Strategy profile \((s_A^*, s_B^*)\) is visualized by the blue lines in the **CENTIPEDE GAME** tree.

\[
\begin{align*}
A & \quad \text{continue} & B & \quad \text{continue} & A & \quad \text{continue} & B & \quad \text{continue} & 5,3 \\
\bullet & \quad \text{stop} & \bullet & \quad \text{stop} & \bullet & \quad \text{stop} & \bullet & \quad \text{stop} \\
1,0 & & 0,2 & & 3,1 & & 2,4 &
\end{align*}
\]
Subgame perfectness in Centipede Game

According to Theorem 14.4, the set of strategy profiles resulting from the backward induction procedure corresponds to the set of subgame perfect Nash equilibria.

Hence, strategy profile \((s^*_A, s^*_B)\) proves to be the unique subgame perfect Nash equilibrium of the CENTIPEDE GAME.
QUESTION: What are the difference between the subgame perfect Nash equilibrium and the other three Nash equilibria of the CENTIPEDE GAME?

ANSWER: The outcome of all Nash equilibria is identical. Indeed, for all Nash equilibria, the game ends immediately after the first round. However, they differ in the players’ behavior at round three and four. The distinctive feature of the subgame perfect equilibrium is that every player would stop the game in every round in which it is her turn to move.
Subgame perfectness in Cake Dividing Game

Exercise: Find all subgame perfect Nash equilibria in the Cake Dividing Game?

▷ To master this exercise we apply again the backward induction procedure.
Backward induction in Cake Dividing Game

**Step 1** Find the best actions of the players who move in subgames of length 1.

Obviously, all subgames $\Gamma(x)$ after history $(x)$, where $0 \leq x \leq 1$ denotes the position of the dividing line chosen by child A, have length of 1.
Backward induction in Cake Dividing Game

Step 1 Find the best actions of the players who move in subgames of length 1.

The best action of child $B$ in subgames $\Gamma(x)$ depends on position $x$. If $x < 0.5$ the best choice is to choose the right piece of cake, if $x > 0.5$ the best choice is to choose the left piece of cake. If $x = 0.5$ both sides are equally good.
Find the best actions of the players who move in subgames of length 1.

To proceed we consider first the case in which child $B$ chooses the left piece of the cake after position $x = 0.5$ is chosen by child $A$ (i.e., in subgame $\Gamma(0.5)$). However, we must bear in mind to repeat the backward induction steps for the case in which child $B$ chooses the right piece in subgame $\Gamma(0.5)$. 

![Diagram of the game]
Backward induction in Cake Dividing Game

Step 2: Taking the best actions in subgames of length 1 as given, find the best actions of the players who move first in subgames of length 2.

Given the just selected best actions of child B in subgames $\Gamma(x)$, the best action of child A is to divide equally the cake.
Backward induction in Cake Dividing

Since the length of the Cake Dividing Game is equal to two, the initial decision node of the game is reached after step 2, and thus the strategies of the players are completely specified.

The first result of the backward induction procedure on the Cake Dividing Game is strategy profile \((s^*_A, s^*_B)\) where

\[
\begin{align*}
    s^*_A(\emptyset) &= 0.5 \\
    s^*_B(x) &= \begin{cases} 
    \text{right} & \text{if } x < 0.5, \\
    \text{left} & \text{otherwise}
    \end{cases}
\end{align*}
\]

holds.
Backward induction in Cake Dividing

NOTE: To figure out all strategy profiles generated by the backward induction procedure, we must also consider the other case in which child \( B \) chooses the right piece of cake after the child \( A \) has divided equally the cake.
Backward induction in Cake Dividing Game

Step 1  Find the best actions of the players who move in the subgames of length 1.

Now, consider the case in which child $B$ chooses the right piece of the cake if position $x = 0.5$ is chosen by child $A$. 
Step 2. Taking the best actions in subgames of length 1 as given, find the best actions of the players who move first in subgames of length 2.

Given the just selected best actions of child $B$ in subgames $\Gamma(x)$, the best action of child $A$ is to divide equally the cake.
Backward induction in Cake Dividing

Since the length of the Cake Dividing Game is equal to two the initial decision node of the game is reached after step 2, and thus the strategies of the players are completely specified.

The second result of the backward induction procedure on the Cake Dividing Game is strategy profile \((s^*_A, s^{**}_B)\) where

\[
\begin{align*}
  s^*_A(\emptyset) &= 0.5 \\
  s^{**}_B(x) &= \begin{cases} 
    \text{right} & \text{if } x \leq 0.5, \\
    \text{left} & \text{otherwise}
  \end{cases}
\end{align*}
\]
Subgame perfectness in Cake Dividing Game

According to Theorem 14.4, the set of strategy profiles resulting from the backward induction procedure corresponds to the set of subgame perfect Nash equilibria.

Hence, strategy profiles \((s_A^*, s_B^*)\) and \((s_A^*, s_B^{**})\) are the subgame perfect Nash equilibrium of the Cake Dividing Game.
Subgame perfectness in Cake Dividing Game

The two subgame perfect Nash equilibria $(s^*_A, s^*_B)$ and $(s^*_A, s^{**}_B)$ can be visualized in the CAKE DIVIDING GAME tree.

Plot of strategy profile $(s^*_A, s^*_B)$:

Plot of strategy profile $(s^*_A, s^{**}_B)$:
QUESTION: Which Nash equilibria of the CAKE DIVIDING GAME are subgame perfect?

ANSWER: We observe that every Nash equilibrium of the CAKE DIVIDING GAME is subgame perfect.