Microeconomics I: Game Theory

Lecture 6: The Indifference Principle

(see Osborne, 2009, Sect 4.3.3, 4.3.4, 4.8, 4.10, 4.11)

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Finding mixed action Nash equilibria

Figuring out directly the set of mixed action Nash equilibria is often a tedious task.

In the following we present two alternative methods for detecting mixed action Nash equilibria:

The first method is based on the players’ best response correspondences, the second one on the indifference principle.
Method of best response correspondence

The method based on players’ best response correspondences (also termed as the two-step method) has been already applied to figure out the set of pure action Nash equilibria.

As in the case of pure action Nash equilibria, we can characterize mixed action Nash equilibria by the players’ best response correspondences.
B.r.c. and mixed action Nash equilibrium

Consider a vNM strategic game $\Gamma := (I, (A_i)_{i \in I}, (\succsim_i)_{i \in I})$ and denote by $B_i$ the best response correspondence of player $i \in I$.

A profile $\alpha^* := (\alpha^*_i)_{i \in I} \in \times_{i \in I} \Delta(A_i)$ of mixed actions is a mixed action Nash equilibrium of $\Gamma$ if and only if, for every player $i \in I$,

$$\alpha^*_i \in B_i(\alpha^*_{-i})$$

holds.
The two-step solution method

The characterization of the mixed Nash equilibrium by players’ best response correspondences suggests following two-step method for detecting mixed action Nash equilibria:

1. Find the best response correspondence $B_i$ for each player $i \in I$.

2. Find all profiles $\alpha^* := (\alpha^*_i)_{i \in I}$ of mixed actions that satisfy $\alpha^*_i \in B_i(\alpha^*_{-i})$ for each player $i \in I$. 
Example: Matching Pennies

In the following, we figure out the mixed action Nash equilibria of the vNM strategic game MATCHING PENNIES by the two step method.
Example: Matching Pennies

To derive the players’ best response correspondences, proceed as follows for each player.

1. Calculate the player’s expected utility if both players randomize.

2. For each possible randomization by the opponents, describe the set of all mixed actions which maximizes the player’s expected utility given this randomization.
Example: Matching Pennies

Suppose player $B$ chooses head with probability $q$ and tail with probability $1 - q$.

If player $A$ chooses head with probability $p$ and tail with probability $1 - p$, then

<table>
<thead>
<tr>
<th>Action Profile</th>
<th>(head,head)</th>
<th>(head,tail)</th>
<th>(tail,head)</th>
<th>(tail,tail)</th>
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</thead>
<tbody>
<tr>
<td>Occurs with probability</td>
<td>$p \cdot q$</td>
<td>$p \cdot (1 - q)$</td>
<td>$(1 - p) \cdot q$</td>
<td>$(1 - p) \cdot (1 - q)$</td>
</tr>
</tbody>
</table>

and $A$’s expected utility is

$$p \cdot q \cdot 1 + p \cdot (1 - q) \cdot (-1) + (1 - p) \cdot q \cdot (-1) + (1 - p) \cdot (1 - q) \cdot 1$$
Example: Matching Pennies

Suppose player $B$ chooses head with probability $q$ and tail with probability $1 - q$.

If player $A$ chooses head with probability $p$ and tail with probability $1 - p$, then her expected utility is

$$U_A(p, q) = pq + p(1 - q)(-1) + (1 - p)q(-1) + (1 - p)(1 - q)1$$

$$= pq - p + pq - q + pq + 1 - q - p + pq$$

$$= 1 - 2q - 2p + 4pq$$

$$= 1 - 2q + 4p(q - \frac{1}{2})$$
Example: Matching Pennies

Suppose player $B$ chooses head with probability $q$ and tail with probability $1 - q$.

If player $A$ chooses head with probability $p$ and tail with probability $1 - p$, then $A$’s expected utility is

$$U_A(p, q) = 1 - 2q + 4p(q - \frac{1}{2}).$$

Hence, the best response $B_A$ is given by

$$B_A(q) = \begin{cases} 
\{0\} & \text{if } q < \frac{1}{2}, \\
[0, 1] & \text{if } q = \frac{1}{2}, \\
\{1\} & \text{if } q > \frac{1}{2}.
\end{cases}$$
Example: Matching Pennies

Plot of $A$’s best response correspondence

$B_A(q)$
Example: Matching Pennies

Suppose player $A$ chooses head with probability $p$ and tail with probability $1 - p$.

If player $B$ chooses head with probability $q$ and tail with probability $1 - q$, then

- Action profile (head,head) occurs with probability $pq$.
- Action profile (head,tail) occurs with probability $p(1-q)$.
- Action profile (tail,head) occurs with probability $(1-p)q$.
- Action profile (tail,tail) occurs with probability $(1-p)(1-q)$.

Player B’s expected utility is:

$$= pq \cdot (-1) + p(1-q) \cdot 1 + (1-p)q \cdot 1 + (1-p)(1-q) \cdot (-1)$$
Example: Matching Pennies

Suppose player $A$ choose head with probability $p$ and tail with probability $1 - p$.

If player $B$ chooses head with probability $q$ and tail with probability $1 - q$, then $B$’s expected utility is

\[
U_B(p, q) = pq(-1) + p(1 - q)1 + (1 - p)q1 + (1 - p)(1 - q)(-1)
\]

\[
= -pq + p - pq + q - pq - 1 + q + p - pq
\]

\[
= -1 + 2p + 2q - 4pq
\]

\[
= -1 + 2p + 4q\left(\frac{1}{2} - p\right)
\]
Example: Matching Pennies

Suppose player A choose head with probability $p$ and tail with probability $1 - p$.

If player B chooses head with probability $q$ and tail with probability $1 - q$, then B’s expected utility is

$$U_B(p, q) = -1 + 2p + 4q\left(\frac{1}{2} - p\right).$$

Hence, the best response $B_B$ is given by

$$B_B(p) = \begin{cases} 
\{1\} & \text{if } p < \frac{1}{2}, \\
[0, 1] & \text{if } p = \frac{1}{2}, \\
\{0\} & \text{if } p > \frac{1}{2}. 
\end{cases}$$
Example: Matching Pennies

Graph of $B$’s best response correspondence

$B_B(p)$
Example: Matching Pennies

The best response correspondences of $A$ and $B$ in one figure.
Example: Matching Pennies

The plot of the best response correspondences of $A$ and $B$ suggests that the mixed action

$$(((p, 1-p), (q, 1-q)) = ((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$$

is a Nash equilibrium of Matching Pennies.

Indeed, the best response correspondences of $A$ and $B$ satisfy

$$\left(\frac{1}{2}, \frac{1}{2}\right) \in B_A\left(\frac{1}{2}, \frac{1}{2}\right) \quad \text{and} \quad \left(\frac{1}{2}, \frac{1}{2}\right) \in B_B\left(\frac{1}{2}, \frac{1}{2}\right).$$
Exercise: Battle of Sexes

Player A

<table>
<thead>
<tr>
<th></th>
<th>Bach</th>
<th>Stravinsky</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>1−p</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

Player B

<table>
<thead>
<tr>
<th></th>
<th>q</th>
<th>1−q</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bach</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Stravinsky</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

ExerciCe: For above vNM strategic game Battle of Sexes, determine the expected utility of A and B and their best response correspondences! Plot them in one figure. Find the set of all mixed action Nash equilibria.
Exercise: Battle of Sexes

Graph of the best response correspondences of $A$ and $B$
Finding mixed action Nash equilibria

Another method to figure out the set of mixed action Nash equilibria is to apply the so-called *indifference principle*, which is stated next.

The indifference principle is an alternative characterization of mixed action Nash equilibria.

Before stating this principle some useful notation is introduced.
Mixed action profiles containing pure actions

Consider vNM strategic game \( \Gamma := (I, (A_i)_{i \in I}, (\succ_i)_{i \in I}) \) and let \( a_i \in A_i \) be a definite action of player \( i \).

The mixed action profile \((a_i, \alpha_{-i}) \in \times_{i \in I} \Delta(A_i)\) means that

- player \( i \) chooses the definite action (or synonymously, the degenerated mixed action) \( a_i \),
- every player \( j \) different to \( i \) chooses the mixed action \( \alpha_j \).
Mixed action profiles containing pure actions

The expected utility of player $i$ obtained by mixed action profile $(a_i, \alpha_{-i})$ is

$$U_i(a_i, \alpha_{-i}) = \sum_{a' \in A} \left( \prod_{j \in I} \alpha_j(a'_j) \right) u_i(a'_i, a'_j)$$

where the second row follows from the fact that $\alpha_i$ is the degenerated lottery that ascribes probability of 1 to action $a_i$. 
Indifference principle

Theorem 6.1

Consider a vNM strategic game \( \Gamma := (I, (A_i)_{i \in I}, (\succeq_i)_{i \in I}) \).

A profile \((\alpha_i^*)_{i \in I}\) of mixed actions is a Nash equilibrium in mixed actions of \(\Gamma\) if and only if, for each player \(i \in I\), the following two conditions are satisfied:

1. \(U_i(a_i, \alpha_{-i}^*) = U_i(a'_i, \alpha_{-i}^*)\) holds for every actions \(a_i, a'_i \in A_i\) to which \(\alpha_i^*\) assigns positive probability,

2. \(U_i(a_i, \alpha_{-i}^*) \geq U_i(a'_i, \alpha_{-i}^*)\) holds for every action \(a_i \in A_i\) to which \(\alpha_i^*\) assigns positive probability and for every action \(a'_i \in A_i\) to which \(\alpha_i^*\) assigns zero probability.
Indifference principle

The indifference principle states that a profile $\alpha^* := (\alpha_i^*)$ of mixed actions is a Nash equilibrium in mixed actions if and only if, for each player $i \in I$,

1. all definite actions realizable under mixed action $\alpha_i$ yield the same expected payoff for player $i$,

2. all definite actions not realizable under mixed action $\alpha_i$ yield an expected payoff for player $i$ that does not exceed that of definite actions realizable under mixed action $\alpha_i$.
Finding all mixed action Nash equilibria

The indifference principle opens up the following method to find all mixed actions Nash equilibria of a vNM strategic game.

1. For each player $i \in I$, choose a non-empty subset $S_i$ of $A_i$.
2. Check whether there is a mixed action profile $(\alpha_i)_{i \in I}$ where
   (1) $\text{supp}(\alpha_i) = S_i$ holds for every player $i \in I$.
   (2) the two conditions of the indifference principle hold for every player $i \in I$.
3. Repeat the analysis for every collection of non-empty subsets of the players’ sets of actions.
Example: Matching Pennies

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<thead>
<tr>
<th></th>
<th>Player B</th>
<th></th>
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</thead>
<tbody>
<tr>
<td></td>
<td>q</td>
<td>1−q</td>
<td></td>
</tr>
<tr>
<td>Head</td>
<td>1,−1</td>
<td>−1,1</td>
<td></td>
</tr>
<tr>
<td>Tail</td>
<td>−1,1</td>
<td>1,−1</td>
<td></td>
</tr>
</tbody>
</table>

**QUESTION:** Figure out all mixed action Nash equilibria of the vNM strategic game MATCHING PENNIES by the procedure based on the indifference principle.
Example: Matching Pennies

The following combinations of supports of mixed actions must be checked.

Support of player $B$’s mixed action

<table>
<thead>
<tr>
<th></th>
<th>{H}</th>
<th>{T}</th>
<th>{H,T}</th>
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<tbody>
<tr>
<td>{H}</td>
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<tr>
<td>{T}</td>
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<td></td>
</tr>
<tr>
<td>{H,T}</td>
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</table>

Support of player $A$’s mixed action

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<tr>
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<th>{H,T}</th>
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<tbody>
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<td></td>
</tr>
<tr>
<td>{T}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{H,T}</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example: Matching Pennies

CASE: The supports of mixed actions $\alpha_A$ and $\alpha_B$ are $\text{supp}_A(\alpha_A) := \{\text{Head}\}$ and $\text{supp}_B(\alpha_B) := \{\text{Head}\}$, respectively.

REASONING: Obviously, $\alpha_A = \text{Head}$ and $\alpha_B = \text{Head}$ apply.
Since
\[
U_B(\text{Head}, \text{Head}) = -1 < 1 = U_B(\text{Head}, \text{Tail})
\]
holds, condition 2 of the indifference principle is violated for player $B$.

Hence, no mixed action Nash equilibrium with such supports exists.
Example: Matching Pennies

CASE: The supports $(\text{supp}_A(\alpha_A), \text{supp}_B(\alpha_B))$ of mixed actions $\alpha_A$ and $\alpha_B$ belong to one of the combinations ($\{\text{Head}\}, \{\text{Tail}\}$), ($\{\text{Tail}\}, \{\text{Head}\}$) or ($\{\text{Tail}\}, \{\text{Tail}\}$).

REASONING: Due to arguments similar to that of the previous case condition 2 of the indifference principle is violated by one player.

Hence, no mixed action Nash equilibrium with such supports exists.
Example: Matching Pennies

CASE: The supports of mixed actions $\alpha_A$ and $\alpha_B$ are

$\text{supp}_A(\alpha_A) := \{\text{Head}, \text{Tail}\}$ and $\text{supp}_B(\alpha_B) := \{\text{Head}\}$, resp.

REASONING: Obviously, $0 < p < 1$ and $\alpha_B = \text{Head}$ apply. Since

$$U_A(\text{Head, Head}) = 1 > -1 = U_A(\text{Tail, Head})$$

holds, condition 1 of the indifference principle is violated for player $A$.

Hence, no mixed action Nash equilibrium with such supports exists.
Example: Matching Pennies

CASE: The supports \((\text{supp}_A(\alpha_A), \text{supp}_B(\alpha_B))\) of mixed actions \(\alpha_A\) and \(\alpha_B\) are either \((\{\text{Head, Tail}\}, \{\text{Tail}\})\), \((\{\text{Tail}\}, \{\text{Head, Tail}\})\) or \((\{\text{Tail}\}, \{\text{Head, Tail}\})\).

REASONING: Due to arguments similar to that of the previous case condition \(^1\) of the indifference principle is violated by one player.

Hence, no mixed action Nash equilibrium with such supports exists.
**Example: Matching Pennies**

**CASE:** The supports of mixed actions $\alpha_A$ and $\alpha_B$ are

$$\text{supp}_A(\alpha_A) := \{\text{Head, Tail}\}$$

and

$$\text{supp}_B(\alpha_B) := \{\text{Head, Tail}\},$$

respectively.

**REASONING:** For player $A$, condition 1 of the indifference principle implies that

$$U_A(\text{Head}, \alpha_B) = U_A(\text{Tail}, \alpha_B),$$

or equivalently,

$$q \cdot 1 + (1 - q) \cdot (-1) = q \cdot (-1) + (1 - q) \cdot 1,$$

which results in $q = \frac{1}{2}$. 
Example: Matching Pennies

CASE: The supports of mixed actions $\alpha_A$ and $\alpha_B$ are
$$\text{supp}_A(\alpha_A) := \{\text{Head, Tail}\} \text{ and supp}_B(\alpha_B) := \{\text{Head, Tail}\},$$
respectively.

REASONING: For player $B$, condition 1 of the indifference principle implies that
$$U_B(\alpha_A, \text{Head}) = U_B(\alpha_A, \text{Tail}),$$
or equivalently,
$$p \cdot (-1) + (1 - p) \cdot 1 = p \cdot 1 + (1 - p) \cdot (-1),$$
which results in $p = \frac{1}{2}$. 
Example: Matching Pennies

CASE: The supports of mixed actions $\alpha_A$ and $\alpha_B$ are
$\text{supp}_A(\alpha_A) := \{\text{Head, Tail}\}$ and $\text{supp}_B(\alpha_B) := \{\text{Head, Tail}\}$, respectively.

REASONING: Summing up the previous discussion, the mixed action profile

$((\alpha_A(\text{Head}), \alpha_A(\text{Tail})), (\alpha_B(\text{Head}), \alpha_B(\text{Tail}))) := ((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$

is a mixed action Nash equilibrium.
Example: Matching Pennies

The following combinations of supports of mixed actions are consistent with a mixed Nash equilibrium.

<table>
<thead>
<tr>
<th>Support of player A’s mixed action</th>
<th>{H}</th>
<th>{T}</th>
<th>{H,T}</th>
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</thead>
<tbody>
<tr>
<td>{H}</td>
<td>×</td>
<td>×</td>
<td>×</td>
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<tr>
<td>{T}</td>
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<td>×</td>
<td>×</td>
</tr>
<tr>
<td>{H,T}</td>
<td>×</td>
<td>×</td>
<td>(((\frac{1}{2}, \frac{1}{2})), ((\frac{1}{2}, \frac{1}{2})))</td>
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</tbody>
</table>
**Exercise: Rock Paper Scissors**

<table>
<thead>
<tr>
<th></th>
<th>Player A</th>
<th></th>
<th>Player B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p_R$</td>
<td>$q_R$</td>
<td>Rock</td>
</tr>
<tr>
<td></td>
<td>$p_P$</td>
<td>$q_P$</td>
<td>Paper</td>
</tr>
<tr>
<td></td>
<td>$p_S$</td>
<td>$q_S$</td>
<td>Scissors</td>
</tr>
<tr>
<td>Rock</td>
<td></td>
<td></td>
<td>0,0</td>
</tr>
<tr>
<td>Paper</td>
<td></td>
<td></td>
<td>-1,1</td>
</tr>
<tr>
<td>Scissors</td>
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<td></td>
<td>1,-1</td>
</tr>
<tr>
<td>Paper</td>
<td></td>
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<td>1,-1</td>
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<tr>
<td>Scissors</td>
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<td>0,0</td>
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</tbody>
</table>

**QUESTION:** Figure out all mixed action Nash equilibria of the vNM strategic game ROCK PAPER SCISSORS by the procedure based on the indifference principle.
Exercise: Rock Paper Scissors

The following combinations of supports of mixed actions must be checked.

<table>
<thead>
<tr>
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<tr>
<td>({S})</td>
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<tr>
<td>({R,P})</td>
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<td>({R,S})</td>
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<td>({P,S})</td>
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<tr>
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